

Field-line helicity and adapted polynomials in vortex knots dynamics

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ABSTRACT. The spontaneous decay of tangles of vortex filaments in a turbulent fluid can be described as a topological simplification process that brings a complex network of vortex knots and links to simplify by a finite sequence of reconnections towards an ensemble of unknotted unlinked loops. Here we discuss the key topological stages of this process, and show that a generic vortex cascade process can be analysed by the application of adapted knot polynomials derived from the field-line helicity of fluid mechanics. Since this quantity plays a crucial role in the computation of the adapted polynomials, we discuss some of its key properties in connection with the Wilson loop operator of field theory. To illustrate the use of these polynomials we refer to the simulated time evolution of three interlinked vortices forming Borromean rings and compare the simulation results with the numerical values provided by the polynomial computation of the simplifying pathways interpreted as geodesic flows in an appropriately defined knot polynomial space.

1. Topological cascade of vortex knots

In recent years work on the formation and evolution of complex vortex tangles in classical and quantum turbulence has grown considerably. This has led to the study and the identification of complex networks of elongated structures in the shape of vortex filaments and tubes [31, 33, 39]. Detailed analysis of the dynamical, geometric and topological properties associated with vortex evolution has shown that once produced these filaments may get highly entangled forming long knots and links. Under mutual interaction these vortices tend to reconnect, change topology and by gradual simplification decay, first forming simpler knots, and then an ensemble of small, unknotted, unlinked vortex rings [4, 22, 23, 43, 44].

From direct numerical simulations of vortex tangles produced in superfluid helium and condensates vortex filaments appear as very thin loops that are highly localised in space (see Figure 1). This allows a very accurate identification of their space configuration, and by the analysis of the experimental and simulation data the possibility to study the morphological properties given by their geometry and topology. This approach, initiated by the work of Barenghi and collaborators [6], has led to the current use of methods and tools from classical differential geometry and standard knot theory, including the recent use of the Alexander polynomial, to detect complex knots and links formed during evolution [5, 12]. By applying

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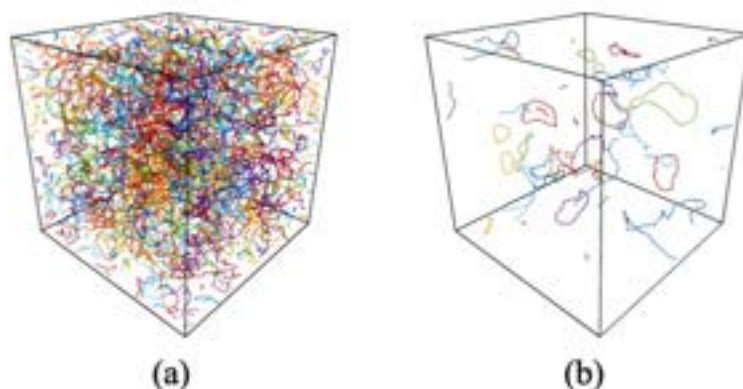


FIGURE 1. Snapshots of the evolution of a complex tangle of quantum vortex filaments produced and governed by the numerical implementation of the Gross-Pitaevskii equation in a periodic box. Snapshots taken at simulation time (a) $t = 12$ and (b) $t = 105$ show the gradual disintegration of an initial network of filaments to a collection of isolated unknotted, unlinked vortex loops (adapted from [39]).

these methods to analyse superfluid turbulent flows a wide spectrum of different knots and links has been found following their formation, simplification and gradual disintegration in time (see Figure 2).

The evolutionary decay that brings large-scale complex knots to small-scale unknots is considered a generic phenomenon common to classical and quantum turbulence, and it can be described as a topological cascade process governed by a time sequence of step-wise unlinking reconnections that occur locally on vortex strands. The topological analysis of this cascade process represents a central, new topic of modern mathematical research at the crossroads of physical knot theory and fluid mechanics [25, 35]. A simple example of topological cascade is the sequence of the unlinking process observed in laboratory experiments and in superfluid simulations (see Figure 3) that brings a trefoil vortex knot T to form a Hopf link H , a single unknot U_1 , and a pair of unlinked, unknotted loops U_2 , before final dissipation (in classical fluids) or evaporation (in quantum fluids) [22, 44]. This topological

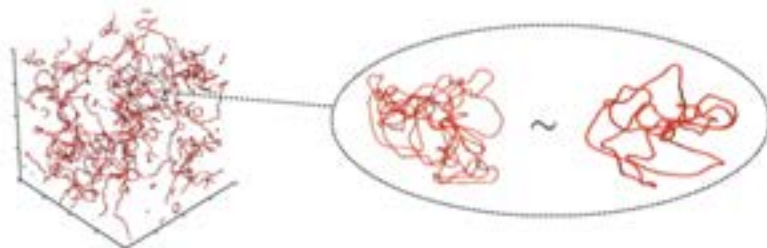


FIGURE 2. Numerical identification, isolation and detection of a complex vortex knot extracted from the instantaneous production of a vortex tangle in superfluid turbulence (adapted from [12]).

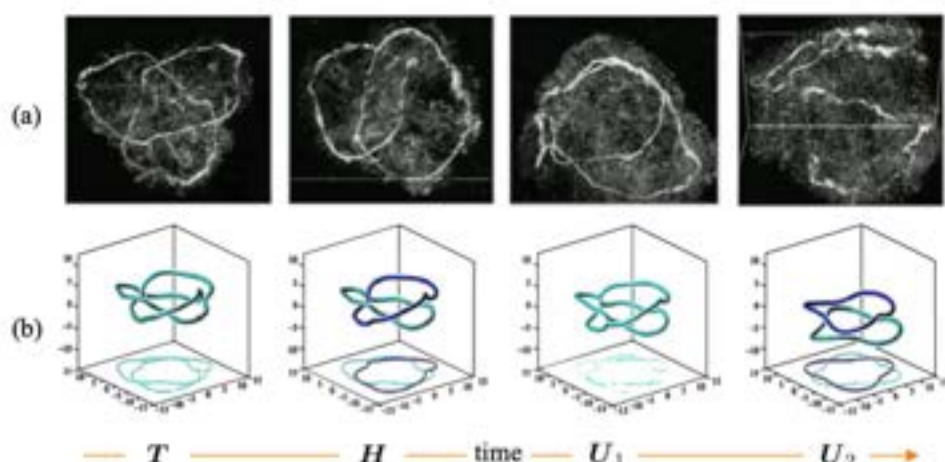


FIGURE 3. (a) Laboratory experiment and (b) numerical simulation of the cascade of a trefoil vortex knot T to a Hopf link H , a single unknot U_1 , and a pair of unlinked, unknotted loops U_2 (adapted from [22] and [44], respectively).

cascade is represented by the time sequence

$$(1.1) \quad T \rightarrow H \rightarrow U_1 \rightarrow U_2 .$$

Similar patterns have been found in very different contexts, such as in the recombination process of DNA-catenanes by enzymatic actions in biology [38], and in the re-organisation of helical flux-tubes by magnetic reconnections in magnetohydrodynamics. The sequence above is produced by the occurrence of a single reconnection event at a time that transforms knots to links and vice-versa. If a knot or a link displays high symmetry (a highly improbable event in freely decaying fluid systems) multiple reconnections can take place at different locations at the same time, so that the intermediate step given by a single reconnection event (that for example brings the trefoil knot to form a Hopf link in the sequence (1.1)) is replaced by an instantaneous topological collapse given for example by $T \rightarrow U_1$ or U_2 . Similarly for more complex knots. Current work on the decay of superfluid Borromean rings B unveils [19, 25] that the short sequence above represents only the final part of a much more elaborate decaying process; an example of this is shown in Figure 4, where the most probable decaying pathways (highlighted by the coloured bands) are described by

$$(1.2) \quad B \rightarrow W_- \begin{array}{l} \nearrow (T^L \sqcup U_1) \rightarrow (H_- \sqcup U_1) \searrow \\ \searrow T^L \rightarrow H_- \rightarrow U_1 \nearrow \end{array} U_2 \rightarrow \dots \rightarrow U_N$$

where the suffix denotes positive/negative knot type according to the relative strand orientation (induced by vorticity).

The work of Cooper *et al.* [12] has shown that turbulent tangles are made by far more complex knots than the 6-crossing Borromean rings mentioned here. Taking the highest degree n of the Alexander polynomial as an approximate measure of

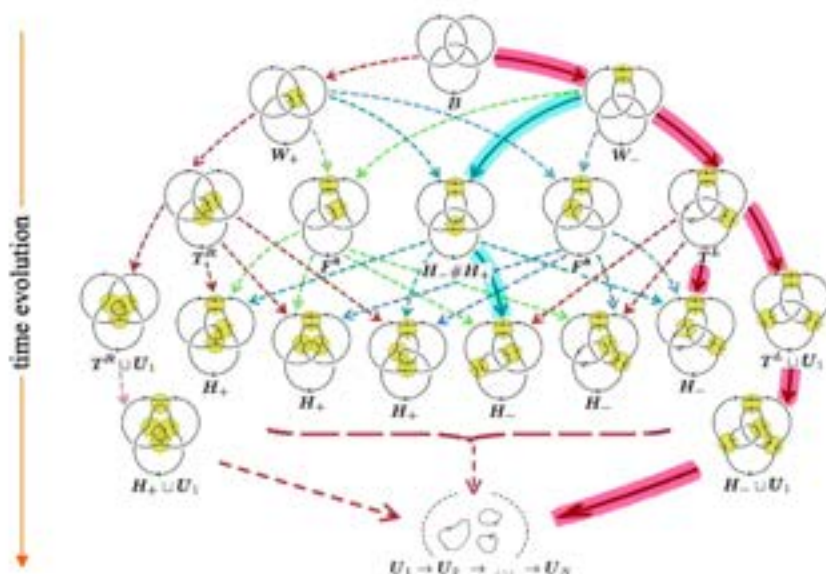


FIGURE 4. Diagram of the expected decaying paths associated with the evolution of quantum Borromean vortex rings; the coloured bands highlight the most probable cascade paths produced by different geometric initial conditions. The suffix denotes the knot/link type determined by the strand orientation induced by vorticity (figure courtesy of Hao Guan, adapted from [19, 25]).

knot complexity, they found the formation of monster knots with $n = \mathcal{O}(100)$, that are well beyond the current capability analysis of standard knot theory. During evolution these knots tend to follow a generic decay path towards the production of simpler knots in agreement with the standard prediction of turbulence theory. Given the high complexity of the tangle, however, local reconnections may produce also knots with a temporary increase of crossing number, including events of inverse cascade from simple to more complex knots [44]. These types of events are rare and in any case transitional, without any lasting effect on the universal, long-time behaviour represented by the overall topological simplification of the dominant decay process. For such complex dynamical systems a detailed sequence of the most probable decay pathway of knots and links is not yet known, but we expect that the use of modern methods based on machine learning techniques may soon provide the required information.

2. Kinetic helicity and Wilson loop

Kinetic helicity is one of the fundamental conserved quantities of ideal vortex dynamics [30]. That is to say its value remains constant in space and time under the fluid flow diffeomorphism φ that governs fluid motion [15]. If $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ denotes the velocity field, function of the vector position $\mathbf{x}$ of a material element in the fluid domain $\mathcal{D} \subseteq \mathbb{R}^3$ and time t , and $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ the vorticity, kinetic helicity

H is defined by the integral

$$(2.1) \quad H = \int \mathbf{u} \cdot \boldsymbol{\omega} \, dV ,$$

where integration is extended over the material volume V of $\boldsymbol{\omega}$ under the condition of no vorticity flux across the bounding surface $S = \partial V$. Note that $\mathbf{u}$ is a global functional of the vorticity, and it is given by the Biot-Savart law

$$(2.2) \quad \mathbf{u}(\mathbf{x}) = \frac{1}{4\pi} \int \frac{\boldsymbol{\omega}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \, dV ,$$

where $\mathbf{x}' \in V$ denotes the location of vorticity; hence, equation (2.1) depends on $\boldsymbol{\omega}$ rather than $\mathbf{u}$. From a field-theoretical viewpoint the divergence-free $\boldsymbol{\omega}$ plays the role of an observable quantity, while $\mathbf{u}$ (which for fluid flows is actually a measurable quantity) plays the role of a gauge potential. A similar situation is given by the magnetic helicity obtained by replacing the vorticity with the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$, and the velocity with the vector potential $\mathbf{A}$. This observation will be important in what follows.

Consider the limiting case of the vorticity field being localised on very thin filaments that form a tangle $\mathcal{T} \equiv \sqcup_i \mathcal{K}_i$ of vortex knots $\mathcal{K}_i$ ($i = 1, \dots, N$). In this case the volume integral (2.1) reduces to the asymptotic form given by a sum of line integrals over the knot length, i.e.

$$(2.3) \quad H = \sum_i \Phi_i \oint_{\mathcal{K}_i} \mathbf{u} \cdot d\mathbf{l} ,$$

where Φ_i denotes the flux of vorticity of the vortex $\mathcal{K}_i$, and $d\mathbf{l}$ the elementary length of the knot. In the case of superfluid and quantum knots vortex strength does not vary during evolution, and it can be taken as a constant and equal for all knots, so that under appropriate time re-scaling we can set $\Phi_i = 1 \, \forall i$. Hence, equation (2.3) simplifies further to the expression given by the field-line helicity

$$(2.4) \quad \bar{H} = \sum_i \oint_{\mathcal{K}_i} \mathbf{u} \cdot d\mathbf{l} ,$$

where overbar denotes appropriate physical re-scaling.

A fundamental result of topological fluid mechanics is that the kinetic helicity of a vortex tangle admits a topological interpretation in terms of linking numbers. By substituting the limiting form of (2.2) into (2.3) the alerted reader can see the emergence of the Gauss linking formula from (2.3). Denoting by Lk_{ij} the (Gauss) linking number between $\mathcal{K}_i$ and $\mathcal{K}_j$, and Sl_i the (Călugăreanu-White) self-linking number of $\mathcal{K}_i$, one can prove [28, 29] that

$$(2.5) \quad H = \sum_i \Phi_i^2 Sl_i + \sum_{i \neq j} \Phi_i \Phi_j Lk_{ij} .$$

Note that this formulation provides a remarkable simplification of the computation of the volume integral (2.3). For a superfluid tangle with $\Phi_i = 1 \, \forall i$, equation (2.5) reduces thus to the simple form

$$(2.6) \quad \bar{H} = \sum_i Sl_i + \sum_{i \neq j} Lk_{ij} ,$$

where the topological character of the helicity is now evident.

As originally pointed out by Călugăreanu [9], it is useful to recall that the topological invariant of the knot $Sl = Sl(\mathcal{K})$ admits decomposition in terms of two geometric quantities of $\mathcal{K}$, namely the writhing number $Wr = Wr(\mathcal{L})$ of the knot centreline $\mathcal{L}$ [16] (not to be confused with the directional writhe of an oriented link diagram), and the total twist number $Tw = Tw(\mathcal{K})$ of the framed knot. Here the writhe denotes a global geometric property of $\mathcal{L}$, and it measures the amount of coiling and distortion of the curve in space. The total twist number Tw admits also a decomposition into two other global geometric quantities [29] given by the normalised total torsion $T = T(\mathcal{L})$ and the intrinsic twist $\mathcal{N} = \mathcal{N}(\mathcal{K})$ of the reference ribbon of the framed knot $\mathcal{K}$.

By taking this decomposition into account we can then equate the field-line integral expression for an isolated, unit strength vortex knot given by (2.4) to its topological and geometric interpretation, so to get

$$(2.7) \quad \oint_{\mathcal{K}} \mathbf{u} \cdot d\mathbf{l} = Sl(\mathcal{K}) = Wr + Tw .$$

As we shall see in Section 3, this relation will be exploited in the derivation and interpretation of the variables of the adapted polynomials of vortex knots.

2.1. Field-line helicity as Wilson line. An interesting connection is provided by the interpretation of the field-line helicity as a Wilson line [40] and the associated operator in the gauge formulation of field theory. This is done by replacing $\mathbf{u}$ with a generic gauge potential $\mathbf{A}$. The fiber bundle formulation of this theory shows [32] that $\mathbf{A}$ plays the role of a connection in the horizontal subspace of the vectors tangent to a curve $\mathcal{C}$ that connects two neighbouring points in the manifold. In vortex dynamics (which is an Abelian theory on $U(1)$) we can identify $\mathcal{C}$ with the vortex centreline $\mathcal{L}$, and in keeping with this analogy state the correspondence

$$(2.8) \quad \mathbf{A} \longleftrightarrow \mathbf{u} \implies \int_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{l} \longleftrightarrow \int_{\mathcal{L}} \mathbf{u} \cdot d\mathbf{l} .$$

In the modern formulation of topological quantum field theory [3] the field-line helicity corresponds to an Abelian Chern-Simons action that is a functional operator that governs the covariant transport under the gauge field $\mathbf{A}$ along a path $\mathcal{C}$ [32, eqs. (10.14)–(10.17)] [11, p. 200, eqs. (3.18)–(3.19)]. Under a diffeomorphism φ this corresponds to set the material derivative of φ

$$(2.9) \quad \frac{d\varphi}{ds} = \hat{\mathbf{t}} \cdot \nabla \varphi = 0 ,$$

where s is arc-length, and $\hat{\mathbf{t}}$ is the unit tangent vector along $\mathcal{C}$. In the most general case the gradient above is replaced by the covariant derivative

$$(2.10) \quad D_{\mu} \varphi = \partial_{\mu} \varphi - i A_{\mu} \varphi$$

(i imaginary unit); equation (2.9) becomes then

$$(2.11) \quad \frac{d\varphi}{ds} + i A_{\mu} \varphi = 0 ,$$

which has a unique solution expressed by the Wilson loop operator

$$(2.12) \quad \varphi(s) = e^{-i \int_{\mathcal{C}} A_{\mu} dx^{\mu}} \varphi_0 .$$

Let us denote by $\xi = \xi(P)$ a Lagrangian parameter along $\mathcal{L}$ that identifies the position of a point P given by $\mathbf{X} \in \mathcal{L}$, with respect to some reference point O given



FIGURE 5. The parallel transport along two different paths $\mathcal{C}$ and $\mathcal{C}'$ admits interpretation in terms of a phase contribution when the two paths are converted to the oriented loop $\mathcal{L}$ (see [40]).

by $X_0 \in \mathcal{L}$; hence $\varphi(s) = \xi \varphi_0$. In the Euclidean vortex context equation (2.11) reduces to the diffeomorphic, material transport of the scalar ξ along $\mathcal{L}$, given by

$$(2.13) \quad \frac{d\xi}{ds} = (\mathbf{u} \cdot \nabla) \xi,$$

or

$$(2.14) \quad \frac{d\xi}{\xi} = (\mathbf{u} \cdot \nabla) ds.$$

Under the ‘adiabatic’ approximation $\nabla_s ds \approx \hat{\mathbf{t}} ds = d\mathbf{l}$, we can directly integrate (2.14) to get

$$(2.15) \quad \xi(P) = e^{\int_{\mathcal{L}} \mathbf{u} \cdot \hat{\mathbf{t}} ds} = e^{\oint_{\mathcal{K}} \mathbf{u} \cdot d\mathbf{l}},$$

which is equivalent to (2.12).

2.2. Properties of the Wilson loop operator. Chern-Simons theory has become the benchmark of a topological quantum field theory, where the physical observable of this theory is given by the Wilson loop [3, 18, 41]. These represent the fundamental states of the theory when computed by the Jones polynomial over knotted paths. A similar approach has been used to construct a topological theory of vortex dynamics [37]. Before making this clear, it is useful to recall some fundamental properties of the Wilson loop operator expressed as field-line helicity.

- *Phase interpretation and holonomy.* The parallel transport of the gauge potential along two different paths (denoted by $\mathcal{C}$ and $\mathcal{C}'$ in Figure 5, left) guarantees the invariant transport of the gauge symmetry from one point to another. When the two paths are converted to form a single loop $\mathcal{L}$ (as in Figure 5, centre), then the loop operator and the area bounded by $\mathcal{L}$ convey some additional information. One immediate interpretation is in terms of phase [10]. If we think of $\mathcal{L}$ as a frame bundle we can associate the holonomy of the loop of frames to its contractibility. The non-triviality of this holonomy is indeed a measure of the geometric (Berry) phase associated with the transport of the potential given by the Wilson loop [32, eq. 10.26]. Since holonomy is a way to lift $\mathcal{L}$ from the base space to the principal bundle of the connection, the Wilson loop accounts for the twisting effect of transporting the information around the loop. To ensure a covariant transport even when $\mathcal{L}$ is a topologically non-trivial knot framed by a smoothly closed ribbon we expect that the phase is always a multiple integer of 2π [20, 41], but of course weird exceptions are possible.

An extension of this geometric interpretation arises when the ambient space $\mathcal{D}$ is multiply connected due to the presence of defects (holes). In this case when $\mathcal{L}$

winds around one (or more) of these defects the holonomy of the loop of frames is no longer contractible, and the phase acquires a topological meaning due to the presence of these obstructions. The relation between the multiple connectedness of the ambient space and the topological phase was originally pointed out by Dirac [13], when he postulated the possible existence of a magnetic monopole. The net effects of the non-trivial topology of the ambient space on the physics was envisaged by Aharonov and Bohm [1], and finally demonstrated to be true by laboratory experiments towards the end of the last century [8].

• *Solid angle interpretation.* An alternative interpretation of the line integral (2.7) comes from the right-hand side of that equality due to Fuller's re-formulation of the writhing number in terms of spherical area. For this consider the spherical curve $\mathcal{L}^*$ on the unit sphere S^2 obtained by the (tantrix) map defined by the unit tangent $\hat{t} : \mathcal{L} \mapsto \mathcal{L}^*$ (see Figure 6a); if A denotes the area in steradians (counted with multiplicity) on S^2 given by the solid angle enclosed by $\mathcal{L}^*$, then we have [17]

$$(2.16) \quad 1 + Wr = \frac{1}{2\pi} A \quad \text{mod } 2.$$

If a unit source point (representing a magnetic monopole) is placed at the centre O of the unit sphere S^2 , the area A becomes a measure of the flux of this source point through $\mathcal{L}^*$ [7, 36]. This interpretation was investigated by Wu & Yang [42], who recognised and corrected the topological non-triviality of Dirac's treatment by using two Euclidean charts (for the north and south hemispheres) to cover S^2 . Since the solid angle subtended by $\mathcal{L}^*$ at O is measured by the writhe of $\mathcal{L}$, there is a direct relation between $Wr(\mathcal{L})$ measured by the spherical area bounded by $\mathcal{L}^*$ and the physical flux through $\mathcal{L}^*$ of the associated Chern class (Figure 6b).

• *Minimal Seifert surface energy.* Another interesting interpretation comes from the application of the Gauss-Bonnet theorem to the Seifert surface spanned by $\mathcal{L}$ (or $\mathcal{K}$). Among the infinitely many orientable Seifert surfaces of $\mathcal{L}$, we can define the surface of minimal geometric area. As shown by considering the time evolution of vortex defects governed by the Gross-Pitaevskii equation [44], application of the Dirichlet principle to the instantaneous distinguished minimal surface detected during the knot evolution demonstrates that the time sequence of these minimal surfaces is a monotonic decreasing function. Since the minimal surface is also an isophase surface of the governing equation, it also carries energy information. Hence, the minimal Seifert surface of fluid knots associated with the field-line helicity becomes also a critical marker for the vortex cascade. Similar considerations apply in the more abstract setting of quantum field theory and string theory, where minimal surfaces (whose area is related to the expectation value of the Wilson loop) may provide information on the fundamental energy state of the system [14, 34].

3. Adapted polynomials for vortex knots

In recent years Liu & Ricca have proved [25, 37] that the standard polynomials of knot theory such as Jones and HOMFLYPT can be derived from the field-line helicity of fluid knots. In the case of the 2-variable HOMFLYPT polynomial $P_K = P_K(a, z)$ in the parameters a and z we have:

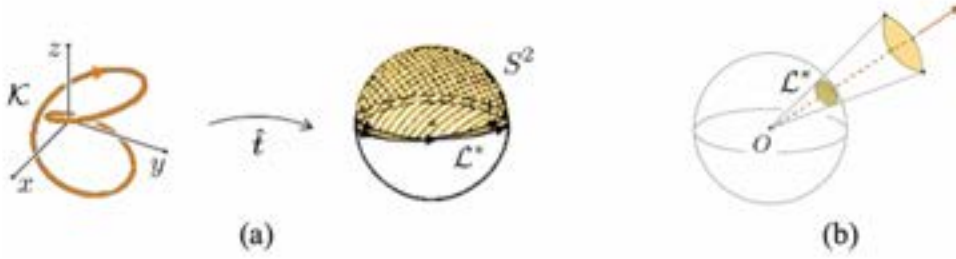


FIGURE 6. (a) The knot centreline is mapped to the spherical curve $\mathcal{L}^*$ on S^2 by the tantrix defined by the unit tangent $\hat{t}$ to $\mathcal{L}$; the writhe of $\mathcal{L}$ is then given by the spherical area $A \bmod 2$: in the case shown $Wr(\mathcal{L}) \approx 1$; (b) The solid angle subtended by $\mathcal{L}^*$ admits interpretation in terms of flux of a source point placed at the centre O of the unit sphere.

THEOREM 3.1 ([26]). Let $H = H(\mathcal{K})$ be the field-line helicity of $\mathcal{K}$. Then

$$(3.1) \quad e^{H(\mathcal{K})} = e^{\oint_{\mathcal{K}} \mathbf{u} \cdot d\mathbf{l}} ,$$

appropriately re-scaled, satisfies (with a plausible statistical hypothesis) the skein relation of the HOMFLYPT polynomial $P = P_{\mathcal{K}}(a, z)$, where $a = a(Tw)$ and $z = z(Wr)$.

The key idea here is that under the transport $\varphi = \varphi(\mathcal{K})$ the vortex knot $\mathcal{K}$ conserves all its properties. Hence, by identifying a and z with the variable ξ of (2.15) a and z satisfy that equation individually. Using (2.7), we thus have

$$(3.2) \quad e^{\oint_{\mathcal{K}} \mathbf{u} \cdot d\mathbf{l}} = e^{Wr + Tw} ,$$

that allows to express a in terms of the total twist Tw , and z as function of the writhing number Wr , both measurable quantities of $\mathcal{K}$; it is possible to prove that

$$(3.3) \quad a = e^{\lambda_r Tw} , \quad z = \frac{e^{4\lambda_\omega Wr} - 1}{e^{2\lambda_\omega Wr}} ,$$

where $\lambda_r \in [0, 1]$ and $\lambda_\omega \in [0, 1]$ represent the uncertainty (assumed to be independent of one another) of the twist and writhe values, respectively.

By specifying the numerical values of Wr and Tw the adapted polynomial becomes then a topological quantifier of vortex dynamics. If, for example, we take the average values $\langle \lambda_\omega \rangle = \langle Wr \rangle = 1/2$ and $\langle \lambda_r \rangle = \langle Tw \rangle = 1/2$, we obtain

$$(3.4) \quad a = e^{1/4} \approx 1.28 , \quad z = e^{1/2} - e^{-1/2} \approx 1.04 .$$

By taking these values for a and z we get a numerical value for the polynomial $P_{\mathcal{K}}(a, z)$. Some of these values, along with the standard HOMFLYPT polynomial, are listed in Table 1 for some elementary knots and links.

Similarly we can get numerical values for the 1-variable Jones polynomial $V_{\mathcal{K}}(t)$. The standard reduction of HOMFLYPT to Jones is given by taking $a = t^{-1}$ and $z = t^{-1/2} - t^{1/2}$, so that $z = a^{1/2} - a^{-1/2}$. By substituting equations (3.3), we get

$$(3.5) \quad \lambda_r Tw = 4\lambda_\omega Wr ,$$

that determines the interdependence between writhe and twist (a fact that is consistent with the helicity constraint in condensates). Taking $a = e^{1/4} \approx 1.28$ we get

TABLE 1. Numerical values of the adapted HOMFLYPT and Jones polynomial for some elementary knots and links; $a = 1.28$, $z = 1.04$, and $t = 0.78$.

Knot/link type	Knot polynomial	Value
Borromean rings B	$P(a, z) = a^{-2}z^{-2} - a^{-2}z^2 + a^2z^{-2} - a^2z^2 - 2z^{-2} + 2z^2 + z^4$ $V(t) = -t^{-3} + 3t^{-2} - 2t^{-1} + 4 - 2t + 3t^2 - t^3$	1.13 4.05
Whitehead link W_+	$P(a, z) = -a^{-1}z^{-1} - a^{-1}z + az^{-1} + 2az + az^3 - a^3z$ $V(t) = t^{3/2}(-1 + t^{-1} - 2t^{-2} + t^{-3} - 2t^{-4} + t^{-5})$	1.59 -2.63
Figure-of-8 knot F^8	$P(a, z) = a^{-2} - 1 - z^2 + a^2$ $V(t) = t^{-2} - t^{-1} + 1 - t + t^2$	0.17 1.19
Trefoil knot T^R	$P(a, z) = 2a^{-2} + a^{-2}z^2 - a^{-4}$ $V(t) = t + t^3 - t^4$	1.51 0.88
Hopf link H_+	$P(a, z) = a^{-1}z + (a^{-1} - a^{-3})z^{-1}$ $V(t) = -t^{1/2}(1 + t^2)$	1.10 -1.13
Unknot U_1	$P(a, z) = 1$ $V(t) = 1$	1.00 1.00
$N \geq 2$ unknots U_N	$P(a, z) = \delta^{N-1} = [(a - a^{-1})/z]^{N-1}$ $V(t) = \delta^{N-1} = (-t^{-1/2} - t^{1/2})^{N-1}$	0.48^{N-1} $(-2.01)^{N-1}$

$t \approx 0.78$, with the numerical values of knots and links shown in the last column of Table 1. Note that the polynomial expressions, and therefore their associated values, are subject to a possible scale change.

4. Topological cascade by geodesic flows in knot polynomial space

As we see from the last column of Table 1 the numerical values of the adapted polynomials do not provide a reliable numerical sequence for the vortex cascade process discussed in the introduction, so we suggest an alternative approach. Following Arnold's idea [2] we propose to interpret the topological untying process in terms of geodesic flows on an appropriate space defined by the adapted knot polynomial. Let $\mathcal{V}_n$ be such a space, whose points (singletons) are the adapted knot polynomials of the knot types produced by the vortex cascade process. To ensure a one-to-one correspondence between knot type and polynomial consider for simplicity the adapted Jones polynomials of the first prime knots with crossing number $c_{\min} \leq 9$. We want to establish a correspondence between the evolutionary flows determined by the vortex cascade and the optimal pathways defined on $\mathcal{V}_n$ (see Figure 7).

A suitable basis (and metric) for $\mathcal{V}_n$ is given by projecting the Jones polynomial $V(\mathcal{K})$ onto the space of standard orthogonal polynomials given, for example, by the Legendre polynomials $\{\mathcal{L}_n(x)\}$ of degree n . Other polynomials such as Hermite's would be equally useful. The orthogonality condition of these polynomials is given by

$$(4.1) \quad \langle \mathcal{L}_n, \mathcal{L}_m \rangle = \int_{-1}^1 \mathcal{L}_n(x) \mathcal{L}_m(x) dx = \frac{2}{2n+1} \delta_{nm}.$$

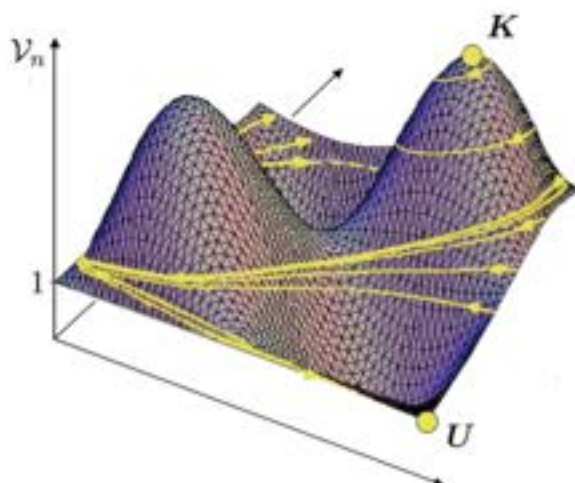


FIGURE 7. Pictorial representation of the evolutionary cascade of a vortex knot K to the unknot U in terms of geodesic flows (coloured lines) on the knot polynomial space V_n visualised as a manifold. The vertical axis is function of the numerical values of the adapted knot polynomial, the horizontal plane is a two-dimensional reduction of the higher-dimensional representation of the phase space.

The standard Legendre polynomial can be reduced to have unit norm by taking

$$(4.2) \quad L_n(x) = \sqrt{\frac{2n+1}{2}} \mathcal{L}_n(x),$$

so that (4.1) gives

$$(4.3) \quad (L_n, L_m) = \int_{-1}^1 L_n(x) L_m(x) dx = \delta_{nm},$$

that provides an orthonormal metric on V_n . A complete basis for $V(K)$ is then given by the family $\{L_n(x)\}$.

For practical purposes, and for the sake of example, let us take $t = e^{\lambda_r T w}$; other values will be equally good and must be ultimately provided by experimental measures, or simulation results. From (3.5) we have $t^{1/2} = e^{2\lambda_\omega W r}$. Noting that different values for t and $\langle W r \rangle$ would simply rescale the results, for simplicity we take $\langle W r \rangle = 1/2$. Since $\lambda_\omega \in [0, 1]$, for $x = t^{1/2}$ we have $x = e^{\lambda_\omega} \in [1, e]$, so that from (4.3) we get

$$(4.4) \quad \int_1^e f[V_n(x)] dx = \frac{2}{e-1} \int_{-1}^1 f[V(x')] dx',$$

in the new variable

$$(4.5) \quad x' = \frac{2}{e-1} x - \frac{1+e}{e-1}.$$

We are now in the position to write

$$(4.6) \quad V(K) \equiv V_K(x') = c_0 L_0(x') + c_1 L_1(x') + \dots + c_n L_n(x'),$$

TABLE 2. Numerical values of the entanglement index $\varepsilon(\mathcal{K})$ for the most probable decay path shown in the chart of Figure 4.

Entanglement index	B	$\rightarrow$	W_-	$\rightarrow$	T^L	$\rightarrow$	H_-	$\rightarrow$	U
$\varepsilon(\mathcal{K})$	1.41		1.14		1.17		0.92		0.00

where $\{c_i\}$ denote the coordinates of each knot type in $\mathcal{V}_n$:

$$(4.7) \quad c_i = \int_{-1}^1 V_{\mathcal{K}}(x') L_i(x') dx' \quad (i = 0, \dots, n).$$

The probability p_i associated with the topological transition between knot/link types is defined by taking

$$(4.8) \quad p_i = \frac{c_i^2}{\sum_{i=1}^n c_i^2} \quad \left(\text{with } \sum_{i=1}^n p_i = 1 \right),$$

and can be used to provide a measure of the occurrence of a given knot/link type in the cascade process; for this we introduce the definition of *entanglement index* $\varepsilon = \varepsilon(\mathcal{K})$ given by

$$(4.9) \quad \varepsilon(\mathcal{K}) = - \sum_{i=1}^n p_i \ln p_i.$$

Direct computation of this quantity for the sequence given by (1.2) provides an almost monotonic decreasing series of numerical values (see Table 2) that represents a more accurate description of the dominant cascade process shown in Figure 4, and this is in good agreement with the results obtained by numerical simulations [19].

5. Concluding remarks and outlook

In this paper we have presented some new results on a novel approach to study the cascade process of vortex knots that occurs in classical and quantum turbulence. This approach is based on the application of standard knot polynomials to physical knots that are dynamical objects subject to a continuous change of shape and topology during their time evolution. Since the cascade process is governed by a topological decay that brings complex knots to gradually simplify till the final production of unknots (see Section 1), the idea is to use adapted standard knot polynomials of knot theory to provide a topological description and quantification of this process. The first step (Section 2) is to relate the knot polynomial variable(s) to the kinetic helicity of vortex dynamics, that for an isolated knot admits a topological interpretation in terms of self-linking number (i.e. writhe and twist). This is done by using the transformation (2.15), which ensures the covariant transport of the polynomial variable obtained by interpreting the Wilson loop operator in the fluid mechanical context. This operator encapsulates information about phase holonomy, writhe, solid angle and Seifert surface energy. Then it is possible to prove (see Theorem 3.1 in Section 3) that the standard knot polynomial can be adapted to *quantify* the knot topology by the physical information carried by writhe and twist helicity associated with the vortex knot. As an example we show in Table 1 some numerical values computed for the adapted HOMFLYPT and Jones

polynomials for simple knots and links. The next step is to provide a reliable description of the observed cascade process. This is done in Section 4 by interpreting the dynamical decay of vortex knots in terms of geodesic flows in an appropriately defined knot polynomial space, whose metric is defined by the numerical values of the adapted polynomial. By computing the relative probability associated with each decaying pathway we introduce the concept of ‘topological occurrence’ to quantify the presence of a specific knot type in the topological cascade; a preliminary account of these results is presented by the values shown in Table 2.

All this seems promising, but much more work is needed. On the one hand the work presented in Section 4 is being currently extended to carry out the analysis and computation of the topological occurrence of all knots and links (including composites) up to 8 crossings [27]. Once completed this will eventually give some stronger support towards a more comprehensive topological treatment of the evolution of fluid systems; in particular, if the measure of topological occurrence introduced by (4.9) will prove useful, we might have a new way to describe physical decaying processes in terms of information entropy (à la Shannon) and standard thermodynamic entropy of fluid mechanics.

On the other hand modern advances in direct numerical simulations will allow to detect and examine the evolution of vortex knots in greater details, so that the implementation and test of the ideas discussed above will be possible. An interesting case, for example, comes from the application of the HOMFLYPT result by Lickorish & Millett [24]:

$$(5.1) \quad P(\mathcal{K}_1 \# \mathcal{K}_2) = P(\mathcal{K}_1)P(\mathcal{K}_2) ,$$

$$(5.2) \quad P(\mathcal{K}_1 \sqcup \mathcal{K}_2) = \delta P(\mathcal{K}_1)P(\mathcal{K}_2) ,$$

where $\#$ denotes connected sum, $\sqcup$ disjoint union of distant knots, and δ defined in the last row of Table 1. For a complex tangle of N connected components, an obvious extension of (5.2) is given by

$$(5.3) \quad P\left(\bigsqcup_{i=1}^N \mathcal{K}_i\right) = \delta^{N-1} \prod_{i=1}^{N-1} P(\mathcal{K}_i) .$$

Moreover, by combining (5.1) and (5.2), we have

$$(5.4) \quad P(\mathcal{K}_1 \sqcup \mathcal{K}_2) = \delta P(\mathcal{K}_1 \# \mathcal{K}_2) ,$$

that suggests a possible interpretation of δ in terms of mutual induction of interacting strands. Since induction is governed by the Biot-Savart law (2.2), one wonders if (to a first approximation) the interaction energy of N well separated vortex knots does indeed scale down as $N - 1$. This and other types of questions offer a new perspective on what may be regarded as a novel topological approach to the evolution of fluid flows and complex dynamical systems.

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