Topological Fluid Dynamics and Knotted Fields

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This article is dedicated to Keith Moffatt, whose living example has been a continuing inspiration for all my scientific life.

Abstract

Topological fluid mechanics provides an ideal setting where concepts and techniques of topology find use and application in the study of the continuous deformation of fluid flows and dynamical systems. It is in this context that knotted physical fields given by vortex flows, magnetic braids, plasma threads and defects turn the mathematics of topology into a vibrant dynamical science, where the transport and exchange of physical properties is subject to the conservation and change of topology.

Key Points

- Continuous deformation
- Kinetic and magnetic helicity
- Tight knots in plasmas
- Knot solutions to Euler equations
- Magnetic braids
- Topological entropy in dynamical systems
- Superfluid vortex knots and turbulent tangles
- Excitable reactions, Seifert surfaces and soap films
- Optical vortex knots
- Hopfions and Skyrmions
- Sedimentation of knotted polymers
- Topological defects in liquid crystals and cosmology

Historical Introduction

Topological fluid dynamics is an active area of research, that deals with problems arising from the study of the continuous deformation of fluid flows and classical fields. Nowadays it encompasses a variety of disciplines ranging from the mathematics of dynamical systems, classical vortex dynamics and magnetohydrodynamics (MHD), to more applied aspects in superfluids, quantum defects, geometric optics, soft matter physics, and others. It relies on the application of the most recent techniques in differential geometry, low dimensional topology (especially knot theory), numerical methods, visiometrics and diagnostics. In this respect it can be truly considered a melting pot of topological concepts and techniques

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that prove useful to tackle open problems in a wide variety of disciplines that involve classical aspects of field theory and applied mathematics.

Topological fluid dynamics (TFD) has its origins in the 'vortex atom theory', envisaged by William Thomson (Lord Kelvin) in the 1860s, that was proposed as a theory of everything *ante litteram*. Relying on Helmholtz's mathematical proof of the permanence of vortex filaments in an ideal fluid, Kelvin postulated that knotted vortices embedded in a microscopic fluid (the so-called *aether*) were the hypothetical, fundamental constituents of matter. The varied and discrete forms of topological knotting and linking of vortices were thus invoked to explain the observed discrete energy spectra of chemical elements. His ambitious programme entailed the mathematical classification of knots and links, that was carried out from scratch by his lifelong collaborator Peter Guthrie Tait in the 20 years that followed. Tait's work led to the correct tabulations of knots and links up to 10 crossings, contributing to the birth of knot theory as a new and emerging branch of topology. As we know, the vortex atom theory was ultimately abandoned, but the uncovered, intimate connection of dynamics and topology was established. New ideas came with the works of Poincaré, Hopf and Courant on one hand, and Dirac, Skyrme and Aharonov on the other, but in the following years the separation between mathematics and physics deepened.

After a period of dormancy it revived with the remarkable contributions by Arnold in the 1960s, and the simultaneous discovery of a new quantity called *helicity*, that is conserved under the ideal evolution of fluid flows. This quantity was derived independently by Woltjer and Moreau in the ideal setting of magnetohydrodynamics (magnetic helicity) and Euler equations (kinetic helicity), respectively. Magnetic helicity is defined by the volume integral of the magnetic field times its vector potential; similarly, kinetic helicity is given by the volume integral of the velocity field times the vorticity. Proof of the time invariance of magnetic helicity is based on the direct application of Faraday's equation, while conservation of kinetic helicity is obtained by using Euler's equation. It is curious to notice that if we regard the integrand abstractly as the scalar product of a vector field and its curl, proof that the integral is an isotopy invariant was already established in a pure mathematical context by Whitehead in 1947. As a result of the scism between pure and applied mathematics occurred before Arnold's work (one of the devastating consequences of the Bourbaki programme) the recognition that the time invariance of helicity was also a topological invariance remained unnoticed for more than twenty years. It is only in 1969 with the work of Moffatt that this correspondence was firmly established and became widely known.

Considering vortex filaments forming links in an incompressible, inviscid fluid Moffatt showed that helicity can be written as a product of vortex circulation (also conserved in ideal conditions) and Gauss' linking number, one of the fundamental topological invariants of knot theory. Because of the fundamental character of this derivation, the result shows a deep connection between the continuous deformation of fluid flows and topology. The implications of this result, however, were not immediate. The importance of helicity in the study of fluid flows and magnetic fields gained momentum only in the 1980s, when the role of coherent, filamentary structures such as vortex and magnetic flux tubes was recognised as a key component in the study of turbulence, dynamo theory and in the energy localisation and transfer in astrophysical flows.

The resurrection of a topological approach to fluid mechanics became reality with the astonishing progress of low dimensional topology, and the outburst of results from knot polynomial invariants (such as Jones, Kauffman, HOMFLYPT, Vassiliev). After a prolonged period of separation which marked a divergence between the formal and rigorous development of topology and the more intuitive and exploratory approach of fluid dynamics, a process of symbiosis between the two fields started to take place more vigorously (see for instance the monograph by Arnold and Khesin, 2021, whose first edition was originally published in 1998, and the collection of papers in Ricca, 2009). A full realisation of the importance of topology in physical science gained further impulse with some landmark experiments, including the confirmation of the Aharonov-Bohm effect in physics, the laboratory production of knotted and linked vortices in water, the topological entanglement of monochromatic waves in free space, accompanied by the continuing advancements in computational visualization and diagnostics of complex flows (see the recent collection of review papers in Ricca and Liu, 2024).

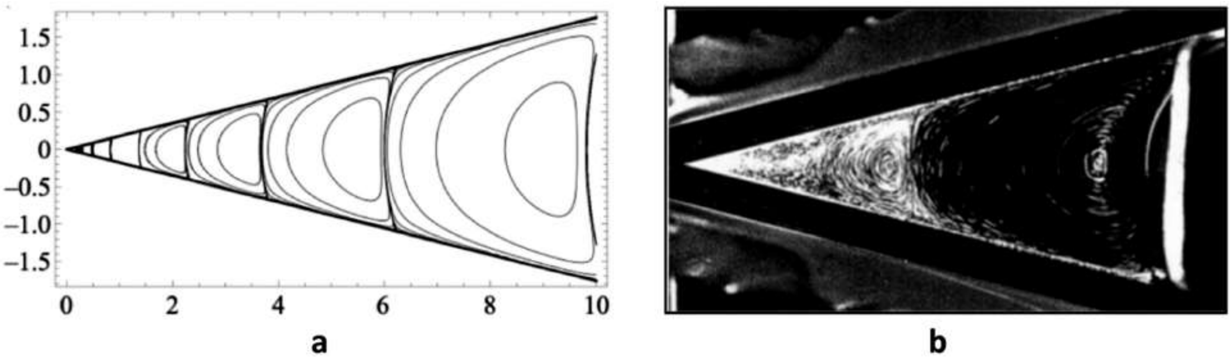

Fig. 1 (a) Moffatt's asymptotic analysis of the formation of counter-rotating eddies in the two-dimensional flow confined in a corner between two planes (adapted from: Moffatt, H.K., 1964. Viscous and resistive eddies near a sharp corner. J. Fluid Mech. 18, 1–18). (b) Experimental verification by Taneda (adapted from: Taneda, S., 1979. Visualization of separating Stokes flows. J. Phys. Soc. Japan 46, 1935–1942).

Topology and Fluid Mechanics

Topological fluid mechanics provides the ideal setting where concepts of topology find their natural playground in the study of the evolution of fluid flows and dynamical systems. It is in this context that vector fields given by lines of vorticity, electric or magnetic fields, or even defects form null points, flux tubes, fluid surfaces or domain walls, providing a physical realization of topological regions. Knots, links and multiply connected surfaces find their natural correspondence with systems of physical interest, where the continuous transport and exchange of dynamical properties and energy are put in relation with the conservation and change of topological properties. It is in this context that a mixture of homotopy techniques, algebraic invariants and global geometric bounds are used to provide information on the transport of time invariant quantities of physical significance, their dissipation and possible formation of singular patterns.

A beautiful example of this symbiosis is shown in **Fig. 1**, that illustrates the behavior of a two-dimensional flow confined in a corner between two planes. As Moffatt's asymptotic analysis predicts (**Fig. 1**a) the flow exhibits the emergence of a geometric sequence of counter-rotating eddies. The experimental verification carried out by Taneda (**Fig. 1**b) demonstrates the physical separation and reattachment of the viscous fluid realised through the formation of cells of decreasing size, whose Hopf index is the topological signature of the flow (see the review paper by Moffatt, 2021).

Vortex structures formed by the motion of a uniform flow in the presence of multiple obstacles offer additional difficulties. By using dynamical system methods Yokoyama and Sakajo (2013) considered the problem of a two-dimensional, incompressible and inviscid flow in a multiply connected domain, devising a procedure to construct structurally stable streamline patterns starting from the basic solution of a flow in a disc of low genus (small number of holes). This method has been developed to anlayse several phenomena that arise in many different contexts, from oceanic and atmospheric flows of environmental interest, to biofluid flows in the human body.

Another interesting and challenging situation is given by a fluid percolating a porous medium. Since the pore-scale arrangement influences the dynamics, here too we expect a hierarchical organisation of the fluid pattern, with a source term represented by the Euler characteristic that constrains the structural evolution. Preliminary evidence seems to suggest that there is a unique relationship between fluid volume, surface area and interfacial curvature on one hand, and domain connectivity on the other (McClure *et al.*, 2020). The challenge here is to encode a discontinuous topological information due to the multiple connectivity of the ambient space with the dynamical description of a continuous fluid process. This kind of analysis has important consequences, especially in relation to the study of other types of fluids, such as foams, suspensions and gels, where the topology of the fluid gets even more fragmented, and the dynamics more complex.

Kinetic and Magnetic Helicity

Kinetic helicity is a conserved quantity of the Euler equations, and it is defined as the volume integral of the inner product of two fundamental fields, namely velocity and vorticity. When vorticity is localised to form a link of vortex tubes of small cross-section and given circulation (also conserved in an ideal fluid) it is possible to express the volume integral as a product of the relative circulations and the Gauss linking number between the tubes. This fundamental result (see again Moffatt, 2021) establishes a deep connection between classical fluid mechanics and topology. If each tube is also knotted in space, then there is an additional contribution to the helicity from the self-linkage of each tube, expressed by the Călugăreanu-White self-linking number (Berger and Fields, 1984; Moffatt and Ricca, 1992), which is a topological invariant of framed knots. More generally, the helicity of a complex tangle of several vortex knots and links is given by the algebraic sum of the products of the linking numbers of the system with their relative circulations which represents, a much easier and computationally advantageous method than the instantaneous numerical integration of vector fields in space.

The computation of linking numbers has further advantages. The Gauss linking number can be computed by the algebraic sum of the apparent signed crossings between knot strands. These are determined straighforwardly from planar projections of the tube centrelines according to some standard sign convention, and since the result is a topological quantity any sufficiently good projection (whose corresponding graph has at most double points) provides the right information. This operation has been implemented in numerical codes, and it proves to be useful. The Călugăreanu-White self-linking number, on the other hand, admits decomposition in terms of two global geometric quantities, the writhing number and the total twist number. The writhing number (or simply 'writhe') is a measure of the folding of the tube centreline in space, and similarly to the Gauss linking number it admits interpretation in terms of algebraic crossing signs averaged over all direction of sights. Since in practice this computation is not feasable, a good estimate of its exact value is given by considering only a finite number of different projections. The total twist number (or simply 'twist') is much harder to compute or estimate, because it is given by the total torsion of the tube axis and the field line winding around that axis (in the knot framing). However, since it is known that in general total torsion contributes only for few percentages, to a first approximation total torsion can be neglected, but internal winding is relevant for energy considerations, so it should be taken into account. In any case, since writhe and twist are directly related to the geometry of the fluid knot in space, both measures provide useful information for the dynamics of vortex knots.

In the presence of viscosity vorticity diffuses, and vortex knots dissipate with time. In this case helicity is not conserved, and linking numbers are no longer invariant. In reality helicity depletion is not that dramatic, and the helicity decay associated with the interaction, reconnection and dissipation of vortex tubes, and the corresponding change of topology, provide useful information on fundamental aspects of vortical flows, especially in relation to the open problem of turbulence. The experimental production of a vortex trefoil knot in

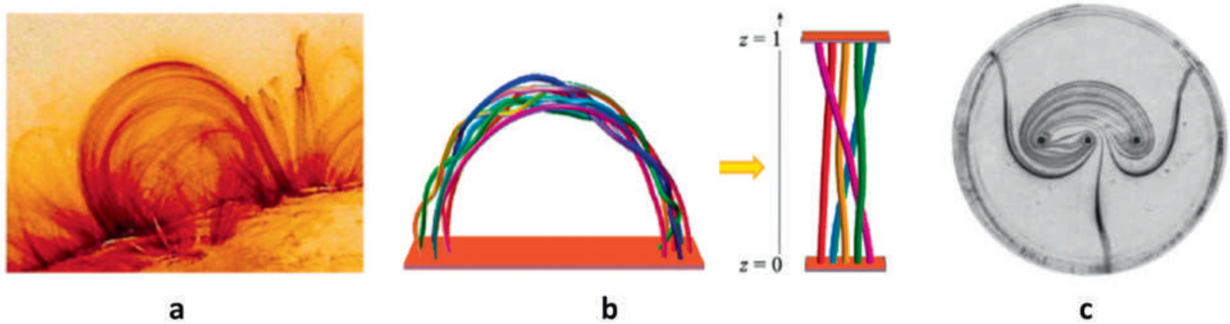

Fig. 2 (a) Ultraviolet image of coronal loops on the Sun taken by the Transition Region & Coronal Explorer (TRACE) satellite in 1998 (from NASA-TRACE mission). (b) The topology of a magnetic braid modelling solar coronal loops can be studied by its representation in terms of a straight braid. (c) Experimental picture of stirring by a pseudo-Anosov protocol applied to three stirrers (adapted from Boyland *et al.*, 2000. Topological fluid mechanics of stirring. J. Fluid Mech. 403, 277–304).

water (Kleckner and Irvine, 2013; see **Fig. 4**a below), with the subsequent evolution and decay into convoluted linked loops and unknots, has provided first-hand evidence of how helicity and writhe get transferred through different physical scales. This has triggered the development of new, sophisticated numerical codes (see, for example, Tao *et al.*, 2021) to validate the information from experiments, and analyse in greater detail the reconnection and re-structuring processes typical of vortex interaction.

In the context of ideal magnetohydrodynamics (MHD) magnetic helicity (similarly to its kinetic counterpart), is simply defined by replacing velocity and vorticity with magnetic field and vector potential. This quantity plays a central role in solar physics and astrophysical flows, where most of the energy released in the outer space comes from the interaction of highly entangled plasma loops that form complex coronal structures in the solar atmosphere (see **Fig. 2**a and section below). With the increased resolution of satellite image acquisition of the coronal region's activity, also energy estimates based on crossing number information and numerical modelling of energy bursts from interacting magnetic flux-tubes and flares have become more accurate and useful, especially in relation to twist helicity contribution. Since plasma physics has also applications in fusion research, work on magnetic helicity has been extended to multiply connected domains, such as the toroidal domain of a tokamak (MacTaggard and Valli, 2019). In this case the connectivity of the ambient space, given for example by the number of holes (Betti number) present in the ambient space, provides additional information for the correction terms of helicity values.

Braids in Magnetohydrodynamics, Dynamical Systems and Mixing

All knots and links can be mapped to a closed braid representative in the same conjugacy class of the given knot and link. The standard representation of a torus knot and link given by the number of strings wrapped around a mathematical torus provides an example of a braid patterns. When dynamical systems display some periodicity (in space or time) it is generally convenient to analyse the topology in terms of its braid pattern. Braids are naturally formed by the orbits of satellites in celestial mechanics, or by the arches of plasma loops in the solar corona (**Fig. 2**a) or, more abstractly, by interpreting the time evolution of the periodic motion of fluid particles in a planar flow as trajectories in a third dimension.

It is always possible to 'cut open' a closed braid on a torus by preserving the topology and considering instead the open braid obtained by displaying the bundle of strings along a vertical axis (see **Fig. 2**b). In this case the vertical coordinate may play the role of space or time, and the topology is accounted by the number of strings and the ordered sequence of over- and under-passes between each pair of strings along the vertical axis to form a 'word'. According to some convention topological information is analysed by some word rules, and can be put in relation with the dynamical properties of the system.

This approach was used by Berger (1993) to determine lower bounds to the free magnetic energy localised in braided plasma loops, that are continuously formed and destroyed in the solar corona by Sun's activity. These loops have foot points rooted in the photosphere and extend over very large distances above the solar surface. By neglecting curvature effects their braid pattern can be stretched along a vertical axis so that standard tools from mathematical braid theory can be applied to study the topology of the field lines. Since about 90% of the total energy released by Sun's activity gets stored in these plasma loops before being released into the outer space through magnetic reconnections, information on their energy contents is of paramount importance for human activity. Berger's estimates matched well observational evidence, so that nowadays braid techniques are often implemented in direct numerical simulations of solar coronal magnetic fields.

By interpreting the vertical axis as a time axis Boyland *et al.* (2000) applied protocols from braid theory to optimize the mixing of two-dimensional fluid flows by the motion of three stirring rods (see **Fig. 2**c, where the rod positions are denoted by the three black dots). Mixing is produced by dragging fluid particles around by the prescribed, automated, eccentric motion of the rods to optimise mixing. If we map each time frame to the vertical axis, the sequence of the rod positions determines a braid pattern in space-time. The authors demonstrated that mixing is not only strictly controlled by the topological complexity of the braid pattern thus produced, but that a particular sequence of braid moves (corresponding to the particular sequence of the rods' motion) have a dramatic effect on the improved mixing efficiency.

This same technique can be applied to many other natural phenomena, from the macroscale dispersion of pollutants to the microscopic transport of bio-chemical elements; this approach has stimulated new research on the role of action groups and maps in braid theory, and the effects of finite-order, reducible, and pseudo-Anosov action maps on the exponential stretching rate of material lines, particle diffusion and topological entropy (Thiffeault, 2022).

Tight Plasma Knots and Steady Solutions to Euler Equations

The search for ideal shapes (from Platonic solids to Plateau surfaces), and the study of their properties, have always interested mathematicians and physicists. A new approach comes from the application of knot theory to magnetohydrodynamics. Consider a magnetic flux tube in the shape of a knot, such as the trefoil knot of **Fig. 3**a. Because of the magnetic field present in the tube, the knot is subject to a Lorentz force that tends to tighten the knot as much as possible like rubber bands under tension. In the ideal setting of magnetic relaxation under the conservation of magnetic volume and flux, the magnetic tension will bring an initially loose knot to shrink gradually towards an end state reaching a tight configuration. Under conservation of volume, the initial free length will decrease at the expense of an increased tube cross-section, and because of the flux conservation the field intensity will also decrease, determining a gradual decrease of the magnetic energy (being this a quadratic functional of the field intensity). In the absence of reconnections the minimum energy state will be dictated by the topological complexity of the knot (see again Moffatt, 2021). For different knot and link types we may therefore expect a spectrum of groundstate energy levels given by the topological spectrum of knots and links. When the minimum state is reached, we expect that the tight configuration attains also an ideal shape (see the collection of papers in Stasiak *et al*., 1998). In this context ropelength (a pure number defined by the ratio of the shortest knot length and the maximal tube cross-sectional radius) provides a useful measure of topological complexity, so that relations between energy levels and ropelengths can be established.

Some results based on the combined use of analytical methods and computational magnetohydrodynamical techniques are shown in **Fig. 3** for three tight plasma equilibria: (a) the trefoil knot, (b) the figure-of-eight knot, and (c) the Borromean rings (Gross *et al*., 2023). Under some simplifying assumptions and using computational tight knot data a discrete spectrum of energy groundstates for the first knots and links has been determined (Ricca and Maggioni, 2014), but work is far from complete and further improvements are expected.

This type of work has interesting implication for other areas of research. In the context of high energy physics, for example, a theory of glueballs was developed using information extrapolated from the tight knot energy spectrum (Buniy *et al*., 2014), and the concept of knot tightening is considered also in relation to the polymeric crystallization, first proposed by de Gennes back in the 1980s (see **Fig. 8**a, and section below).

From a theoretical viewpoint since steady MHD equilibria correspond to steady Euler flows, existence of tight magnetic knot configurations (i.e. MHD equilibria) has implications also for the existence of steady knot solutions to Euler's equations. The search for vortex knot solutions in fluid flows began with the so-called localised induction approximation (LIA), derived from a cut-off of the Biot-Savart induction law. Kida (1981) showed that a class of steady torus knot solutions exist as solitary waves on a vortex filament, and this led to a series of works on LIA cabled knot solutions. Real progress, however, was only done by Enciso and Peralta-Salas (2012) proving the existence of knots as steady solutions to the Euler equations. Explicit solutions in closed, analytical form, however, are still elusive even for the simple configuration of a trefoil knot.

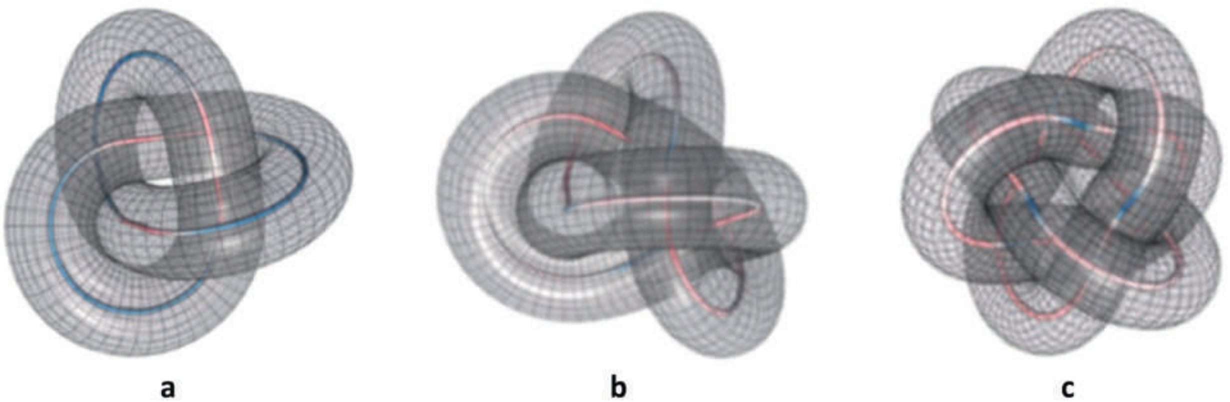

Fig. 3 Three "relaxed" states of magnetic knots representing a stationary point of the magnetic energy: (a) trefoil knot; (b) figure-of-eight knot; (c) Borromean rings (adapted from: Gross *et al*., 2023. Plasma knots. Phys. Lett. A 480, 128986).

Knot Polynomials in Superfluid Turbulence

Helicity, expressed in terms of linking numbers, suffer from the natural limitations of the linking number. As noticed by Maxwell, the Gauss linking number sometimes fails to distinguish unlinked loops (for which this number is evidently zero) from essential links such as the Whitehead link, made by two inseparable loops, or the Borromean rings made by three inseperable rings. This is one of the reasons that pushed mathematicians to look for other invariants able to capture finer topological information. Several types of new invariants were discovered and introduced in the literature, starting from the Alexander knot polynomial of the mid-1920s. The search continued with Conway's work of the 1970s, till the derivation of the Kauffman's brackets, the Jones and HOMFLYPT polynomials of the 1980s, culminating with the classification of Vassiliev of the 1990s and the categorical extensions that followed.

Knot polynomials are rather powerful invariants of low dimensional topology. A knot polynomial is a polynomial of Laurent type in one or more dummy variables, whose coefficients encode some of the properties of the given knot. In a search for more powerful ways to take account of the topology of fluid knots, Liu and Ricca (see the review chapter in Ricca and Liu, 2024) proposed to interpret the polynomial variables in terms of writhe and twist helicity contributions, showing that the so-called field line helicity (a limiting form of helicity for very thin filaments) satisfies the defining skein relation of a knot polynomial. By expressing the polynomial variables in terms of writhe and twist of the fluid knot, it is possible to compute numerical values and measure each physical knot by its polynomial value. This way the new, adapted polynomial is expected to be able to detect the topology of the knot (as standard knot polynomials already do), and quantify physical information in terms of knot strength (vortex circulation or magnetic flux), writhe and twist. The validity and applicability of this proposal remains to be seen, but it has the potential to open new perspectives in the use of new topological methods to study vortical fluid flows.

These ideas may find useful applications primarily in the study of superfluid flows, where highly tangled, localised vortex lines are produced in turbulence. In the milli-Kelvin temperature limit Helium-4 becomes almost perfectly inviscid, and vorticity coalesces into very thin vortex filaments. Standard Alexander polynomials have been used in the diagnostics of direct numerical simulation of superfluid turbulence, demonstrating that complex tangles not only are ubiquitous in the fluid, but they can also achieve an extremely high degree of topological complexity (Cooper *et al.*, 2019). A crude way to estimate this complexity is given by the highest degree of the knot polynomial, and for the case studied the computed values are well beyond standard knot classification. This means that in the superfluid regime vortical flows may form extremely complex knots before decaying through a series of reconnections that entail a topological cascade towards simple, unknotted, unlinked loops before final evaporation. The physical phenomenon is not new, but the novel approach offered by the use of knot polynomials provides new insight into the hard problem of turbulence.

Quantum Knots, Excitable Knots and Seifert Surfaces

The first laboratory production of a knotted vortex in water (**Fig. 4**a; Kleckner and Irvine, 2013) has given impetus to the production of knots and links in a variety of other physical systems, including the first experimental creation of quantum knots in condensates (Hall *et al.*, 2016). The governing equations may have different origin, but they share similar mathematical aspects. For example the order parameter is given by a single, complex wavefunction with the information controlled by a phase map. In

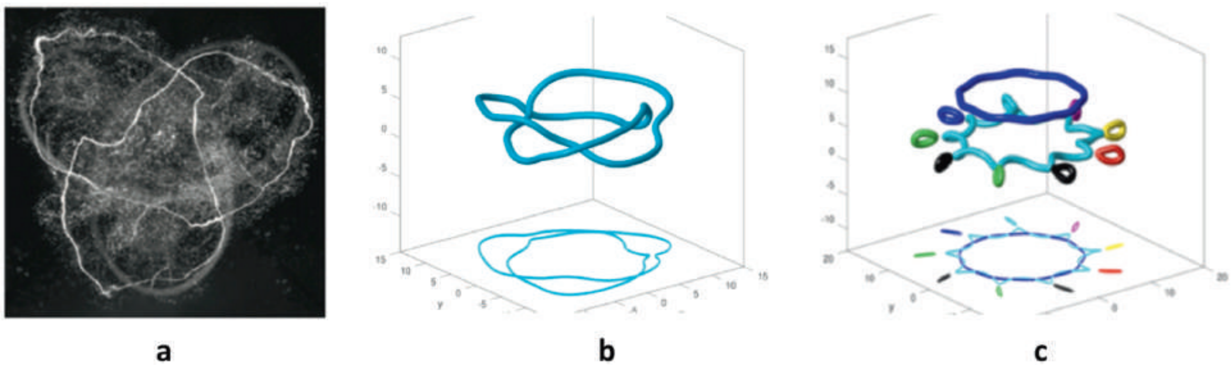

Fig. 4 (a) Laboratory production of a trefoil vortex knot in water (from Kleckner and Irvine, 2013. Creation and dynamics of knotted vortices. Nat. Phys. 9, 253–258). (b, c). Numerical simulation governed by the Gross-Pitaevskii equation: (b) production of a trefoil knot from the interaction of two unknotted, unlinked loop defects, and (c) topological decay of the torus knot (2,9) forming two large, unknotted loops and 9 ringlets (adapted from: Zuccher and Ricca, 2022. Creation of quantum knots and links driven by minimal surfaces. J. Fluid Mech. 942, A8).

this case nodal lines of the wavefunction may determine the presence of defects in the medium: vortex defects in a condensate (a rarefied gas of bosons brought at ultra-low temperature), or organizing centres in an excitable chemical system of reaction-diffusion type are two examples. In such systems defects may carry a quantised circulation, and can be regarded as vortex lines in a classical fluid.

If the system is controlled by a single, global phase map, it is possible to prove (Sumners *et al.*, 2021) that defects forming knots and links of unit charge are subject to the condition of zero-helicity. Since total helicity is given by the algebraic sum of all the linking and self-linking numbers, this global topological feature puts a strong constraint on the existence and stability of these defects. For a single defect in isolation the zero self-linking condition implies that writhe and twist must always sum up to zero. This means that a superposition of an external twist phase must trigger the production of a new defect to balance the presence of the new twist, a scenario that may open new avenues for technological applications. The physical implications of this topological constraint were noticed earlier on by Winfree and Strogatz (1984) studying properties of the Belousov-Zhabotinskii reaction, and it is now a proven fact common also to quantum defects in condensates (**Fig. 4**b,c).

The existence of a global phase map has shifted attention to the study of the orientable surfaces bounded by knots and links. These surfaces, called Seifert surfaces, are not unique, but if we restrict attention to the Seifert surfaces of minimal geometric area, or critical energy, we may gather information of physical relevance. This is a very beautiful area of research at the interface of topology, differential geometry, and physics. A detailed study of some of these properties was carried out for vortex defects whose dynamics is governed by the so-called Gross-Pitaevskii equation. Recent analysis on the evolution of knots and links shows that the associated isophase minimal surface, that carries also physical information on the total energy, keeps decreasing in time as the knot evolves in time (Zuccher and Ricca, 2022). Moreover, since during evolution defects interact and reconnect in an anti-parallel fashion (vortex defects must conserve orientation after a change of topology), writhe remains conserved across a reconnection (Laing *et al.*, 2015). Under the zero-helicity condition this implies the conservation of twist and a smooth re-organisation of the incident Seifert surfaces during the reconnection process. Since orientability is preserved, there is hope to determine new relations between topological bounds dictated by band surgery, reconnection numbers, and the energy associated with the topological cascade.

A similar type of work can be done by considering soap films. As was already noted by Courant in 1940s, curious experimentation with soap films bounded by wire frames of various shapes expose the problems posed by the interplay of topology and physics. A simple experiment can be done considering the continuous defomation of a soap film of minimal area when the bounding wire, given by a single loop initially folded over to form a closed coil, gets gradually unfolded into a large, planar circle. When the wire is folded over, the soap film of minimal area is a Möbius strip (a two-sided surface), and when the wire is gradually deformed to become a planar circle the minimal surface becomes a disc (which is one-sided). The topological transition of the soap film, that occur through a continuous re-arrangement of the fluid surface takes place by a jump of the linking number bewteen the soap film Plateau border and the wire, a very interesting and curious phenomenon (Goldstein *et al.*, 2010). Accurate experimental work shows that the abrupt re-arrangement of the fluid surface takes place by a twisted fold (or cusp) catastrophe, a beautiful example of the interplay of topology and fluid dynamics

Knots and Links in Electromagnetic Fields and Geometric Optics

In the case of electromagnetic fields, the first knotted solution to Maxwell's equations can be probably traced back to Synge, but the first detailed study of the existence and topological properties of linked field lines was carried out by Rañada (1989). By

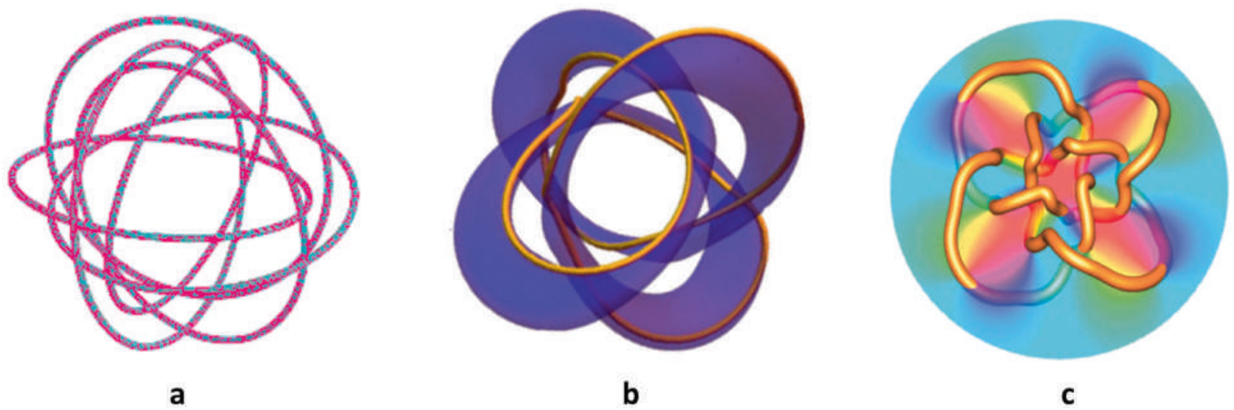

Fig. 5 (a) Pairwise linking of magnetic lines as a model for the ball lightning (adapted from Rañada *et al.*, 2000. Ball lightning as a force-free magnetic knot. Phys. Rev. E 62, 7181–7190). (b) Magnetic line tangent to a nested knotted torus encoded in the level set of a complex scalar field (from Kedia *et al.*, 2016. Weaving knotted vector fields with tunable helicity. Phys. Rev. Lett. 117, 274501). (c) Optical vortex knot 8_{18} propagating from a hologram (from Dennis *et al.*, 2010).

making use of stereographic projections, from the fundamental properties of electric and magnetic fields whose lines of force are orthogonal to each other, it is possible to reconstruct Hopf link solutions. The Hopf index is a measure of the magnetic helicity, and link solutions are classified in homotopy classes.

The discovery of linked solutions led Rañada and collaborators to propose a topological model for the formation and stability of ball lightning (that remains an unexplained natural phenomenon given by a flaming ball produced near the electrical discharge of a normal lightning). The proposed model relies on the creation of electric currents flowing along short-circuited linked streamers of ionised plasma. These streamers behave as highly conducting coils confined to a spherical region of hot air. **Fig. 5**a illustrates a pictorial representation of 6 of such streamers linked pairwise once. The model stirred up some controversy, partly motivated by unsettled questions regarding the stability, and partly by the physics involved. Theoretical support came from arguments put forward by Faddeev and Niemi (1997), who showed that the virial theorem does allow the existence of topologically stable solitons in plasmas that describe knotted and linked flux tubes of magnetic fields.

The question of stable linked magnetic lines is indeed delicate, because after creation the field lines tend to reconnect and disappear rapidly. Over 100 years ago, Harry Bateman introduced a formalism to study a family of null fields defined by vanishing values of certain scalar invariants which act as electric and magnetic Euler potentials. Recently, by taking advantage of this formalism researchers derived null electromagnetic fields as closed electric and magnetic field lines in the shape of knots and links (Kedia *et al.*, 2013). Examples of this construction are shown in **Fig. 5**b and **5**c. By exploiting properties of certain holomorphic functions Bode (2021) provides a rigorous existence result for torus links, showing that not only every pair of such links may arise as a set of closed electric and magnetic field lines, but also that any linking between the link components can be realised as a stable configuration.

Another strand of research emerged by exploiting an algebraic construction of knots and links as singularities in complex hypersurfaces discovered by Brauner in 1928, and popularized by Milnor at the end of the 1960s. By applying this construction to a nodal line of a wavefunction Dennis and collaborators were able to produce a rich zoo of knotted and linked optical vortices in laser light generated by computer-controlled holograms (**Fig. 5**c). Since laser light is a monochromatic radiation, from Maxwell's equations one can consider a single, time-independent, complex scalar component of the electric field that satisfies the Helmholtz equation; approximating this equation by its paraxial form, and by applying the Brauner-Milnor technique, one can create nodal lines of any topological complexity (Dennis *et al.*, 2010; for a detailed account, see also the review chapter by Dennis in Ricca and Liu, 2024).

Hopfions and Skyrmions

Techniques of topological dynamics can be extended to other physical systems that rely on continuous families of curves forming knotted fields in different target spaces (complex or real projective). Topological textures based on maps where the target space is the 2-sphere or the 3-sphere have been studied for a long time. Back in the 1960s Skyrme proposed these textures as localised solitons in high-energy physics (see again Faddeev and Niemi, 1997). Ten years later Trautman proposed a novel description of the static magnetic field of a de-singularized pole in terms of gradients of scalar fields (interpreted as Euler angles), opening a new era for investigations. By defining an Euler-like potential as a (complex) sum of these scalar fields, one can interpret each point on the unit 2-sphere given by these angles as a magnetization potential. One can then apply the inverse Hopf map from the 2-sphere to the 3-sphere to view this magnetization as fibers linked (by stereographic projection) in the standard three-dimensional space. Since each

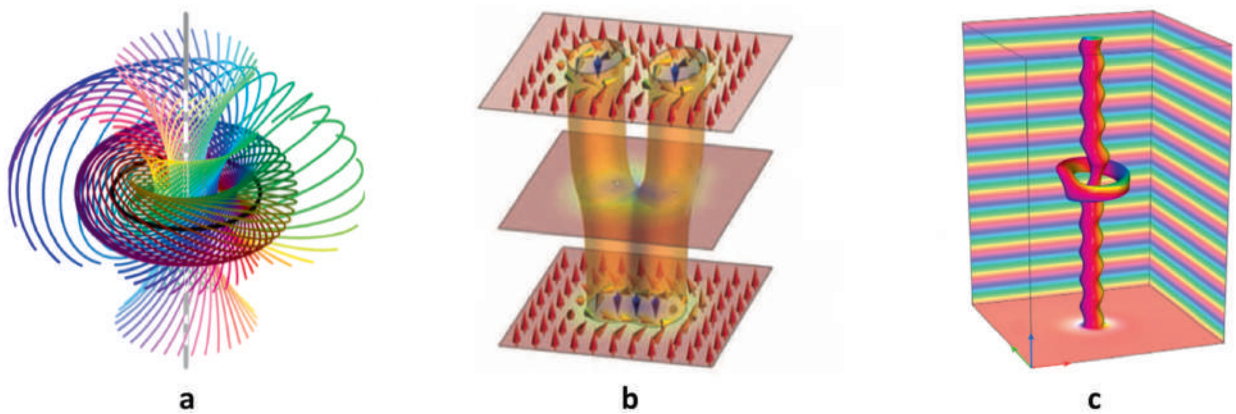

Fig. 6 (a) Hopf fibration given by a texture of successive tori, each of which is made up of a family of Villarceau circles (from Dennis, 2024. Designing knotted fields in light and electromagnetism. In Knotted Fields (R.L. Ricca & X. Liu eds.). Lecture Notes in Mathematics 2344, Springer Nature, Switzerland). (b) Magnetic configuration illustrating the merging of two Skyrmions (adapted from Milde *et al.*, 2023. Unwinding of a Skyrmion lattice by magnetic monopoles. Science 340, 1076–1080). (c) Topological linking between the textures of a Hopfion and a Skyrmion (from Zheng *et al.*, 2023. Magnetic skyrmion braids. Nature Comm. 12, 5316).

point on the 2-sphere correspond to a fiber link on the 3-sphere, the set of points on the 2-sphere give rise to a topological texture for the magnetization potential known as Hopf fibration. An example of such a fibration (or Hopfion) is shown in **Fig. 6**a, where the space is filled by linked field lines defined on nested tori and centered on the vertical grey nodal line of the potential. By construction this texture is a localised 3D object, whose topology is preserved by the protection offered by the energy barrier.

A similar mathematical construction is used to define a Skyrmion, given by a straight braid of magnetised field lines (Manton and Sutcliffe, 2004). Contrary to Hopfions, Skyrmions are almost 2D objects given by a tubular texture of magnetic fibers that spreads out in space. Since magnetization can be expressed by a 2-dimensional unit vector, their topological charge (given by the Hopf index) is measured by the 2-dimensional version of the helicity integral, obtained by replacing the magnetic potential with the magnetisation unit vector. A simple computation of this integral in terms of Euler angles on the unit sphere gives a quantised measure of the winding number of the Skyrmion.

Because these objects are static and topologically stable, they have attracted great interest for technological applications in very many different contexts, from structured condensed matter and liquid crystals to cold atoms physics and quantum computing. Since the first experimental observation of chiral magnets by monochromatic light in cubic crystals (see Sugic *et al.*, 2021) research has diversified, from simple realisations of Skyrmion lattices to the interaction and coupling of Skyrmions and Hopfions (see **Fig. 6**b and c), where more and more complex features of topology and physics are investigated (Zheng *et al.*, 2021).

Topological Defects in Liquid Crystals

Under phase transition systems such as type II superconductors, metamaterials, colloids, liquid crystals, and bio-systems may form a variety of topological defects (such as magnetic vortons and disclinations) that share common characteristics. These defects are homotopically distinct solutions to the differential equations that govern the system. Their emergence is due to the homotopy group that is induced by, and associated with, the specific boundary conditions. Since under continuous deformations of the boundary conditions this characteristic is preserved, their topological type is protected. Homotopy theory, however, does not provide all the necessary information to explore the variety of the topological complexity of such defects. For example, a closed loop made by a disclination may be considered under a certain map whose topological charge depends on how this disclination is closed on itself, its local structure, twisting, writhing, knotting and possible linking with other defect loops. In active nematics, then, some 'topological turbulence' can even occur when the internal energy production counteracts the coarsening by the driving shear flows that extend and split an existing line defect to create new defects by pair creation (Kralj *et al.*, 2023).

A clear understanding of the structural details induced by energy changes cannot be predicted solely by homotopy theory, but can be understood by invoking the analysis of the disclination's structure along the loop. In other words, homotopy theory identifies the knotted fields embedding, but not how to obtain field configurations with desired topological charge. Moreover, when liquid crystals and colloidal ferromagnets interact with surfaces due to various boundary conditions, the topology of structures of these fields interplays with that of the bounding surface, producing new, highly complex systems (Smalyukh, 2020).

Vivid examples of application of topological ideas to soft matter physics are shown in **Fig. 7**. The experimental realization of a handlebody (2-torus) colloidal particle is shown in **Fig. 7**a. When these shapes are dispersed in a liquid crystal, new forms of interactions between these surface topologies and the molecular alignment fields arise due to the conditions at the interface. **Fig. 7**b shows an optical micrograph of a photo-polymerized nematic colloidal trefoil knot with tangential boundary conditions; a two-component Hopf link particle is visualized by optical bright field micrograph in **Fig. 7**c.

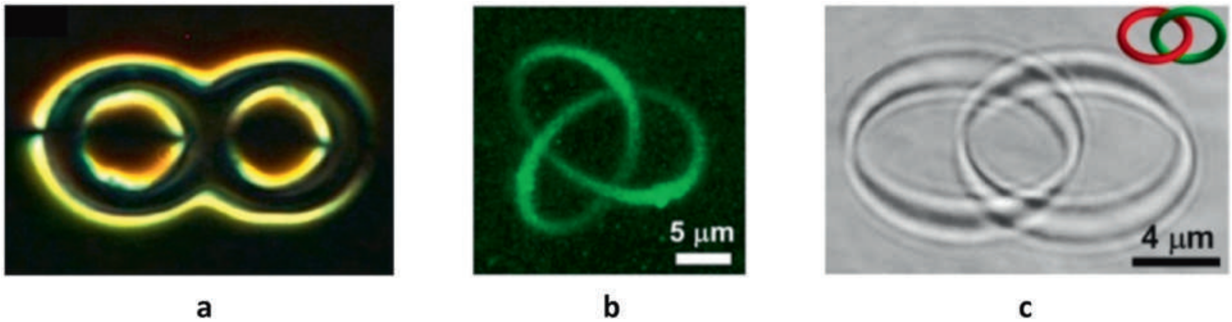

Fig. 7 (a) Polarized optical micrograph of colloidal handlebody with perpendicular surface boundary conditions (from Senyuk *et al.*, 2013. Topological colloids. Nature 493, 200–205). (b) Nematic colloidal trefoil knot shown in nonlinear luminescence image of a particle fabricated by means of the spatially-resolved graphene oxide reduction with femtosecond laser light (from Senyuk *et al.*, 2015. Three-dimensional patterning of solid microstructures through laser reduction of colloidal graphene oxide in liquid-crystalline dispersions. Nat. Commun. 6, 7157). (c) Optical bright field micrograph of a nematic colloidal Hopf link particle with tangential boundary conditions (adapted from Martinez *et al.*, 2015. Linked topological colloids in a nematic host. Proc. Natl. Acad. Sci. 112, 4546–4551).

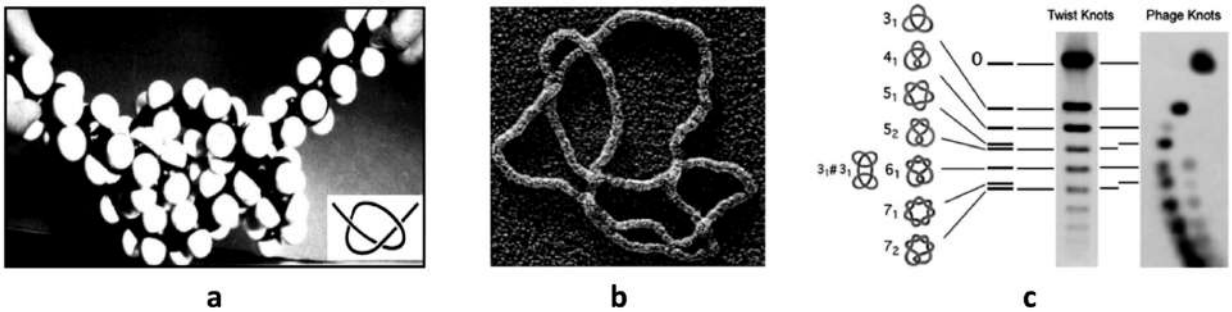

Fig. 8 (a) Tight trefoil knot representation of an aliphatic chain model involving 36–38 carbon atoms in the active region (from de Gennes, 1984). (b) Electron micrograph of a laboratory production of a DNA 6-crossing knot (from Wasserman *et al.*, 1985. Discovery of a predicted DNA knot substantiates a model for site-specific recombination. Science 229, 171–174). (c) Left: geometrical representations of prime knots and the composite knot of two trefoils; right: identification of the specific DNA knots by their position through gel electrophoresis migration (from Arsuaga *et al.*, 2005).

Knotted Polymers, Proteins and DNAs in Fluid Cells

It has long been known that synthetic macromolecules, biopolymers, DNAs, proteins, and even chains of active matter can form knots and links either as artifacts, or by natural mechanisms (Tubiana *et al.*, 2024). Early in the 1980s de Gennes considered the effects of packing and knotting of polymers (**Fig. 7**a) in relation to the crystallization properties in solvents, and nowadays the synthetic production of knotted macromolecules is standard practice. Remarkably, even knotted DNAs have been produced in vitro, and have been observed in organisms and viruses in vivo as well (see **Fig. 7**b).

It is a known mathematical and physical fact that length and confinement favour knotting probability, and because conformational properties are strictly related to functions, the presence of natural knotting and linking raises many fundamental questions in relation to their role. Knotting can preclude transcription and duplication or slow down the passage through pores (a mechanism that viruses can use to their own advantage). Conformational and topological properties of these filamentary structures are intimately related to the hydrodynamics of the fluid medium in which they are dispersed. Think for example of the sedimentation and electrophoretic motility, or the ability to percolate through porous membranes. An example is shown in **Fig. 7**c, where the discrete distribution of the sedimentation of knotted DNAs in solution provides a clue about their packing, topological complexity, and hydrodynamic behaviour.

Since hydrodynamic properties are related to the conformation and the hydrodynamic characteristics of the wet surface of the knotted polymer, there is a call for a detailed study of the interplay of geometry, topology and hydrodynamic drag in complex flows, where not only viscosity plays a role, but also electric forces, salinity and other features play a role (Arsuaga *et al.*, 2005). The presence of holes (Betti number) in confining membranes, then, may provide additional topological complexity to the governing equations due to the multiple connectedness of the fluid domain. Other open aspects regard the understanding of the untying mechanisms performed on knotted strands, and the role of the so-called topological friction invoked to account for the dynamical difference between non-entangled and self-entangled chains in various settings.

As for active fluids, the presence of filaments, myosin motors and passive crosslinks in gel, for example, provides an innovative playground for designing new topological states of matter owing to the broken time-reversal and chiral symmetry. In this case the Chern class of topological invariants provides useful information on the number and direction of the chiral states propagating along the interface of two topological states. More recent developments in topological active matter regard the relation between these topological invariants and the non-Hermiticity property of the fluid system subject to a potential gain and loss of energy (Shankar *et al.*, 2022). Ongoing work will provide new information for the design of novel, smart and reconfigurable materials, where topology and fluid mechanics are closely interconnected.

Topological Defects in Cosmology

The topological classification of dislocations in liquid crystals have applications also in the study of cosmological phase transitions hypothesized by the Big Bang theory. Following the pioneering work of Kibble in the late 1970s, the formation and interaction of massive cosmic strings are envisaged as the cause for the symmetry-breaking phenomena hypothesized by cosmologists in the earliest moments of the universe's evolution. In this context the homotopy groups associated with the vacuum manifold are accounted for the existence of various types of defects, such as domain walls, cosmic strings and monopoles. (Davis and Brandenberger, 1995).

The existence of strings, in particular, would justify the multiple connectedness of the vacuum manifold, where the specificity of its connected components would be determined by the stability properties of these primary objects. According to the different scenarios of phase transitions, we may have strings arising as boundaries of domain walls, or as a result of monopole formation.

The expected physical properties observed would be derived from the exact sequence of the defining homotopy groups. From the newly developed superstring theory of the XXI century, we have now a novel zoo of topological defects, including D-strings, vortons, and higher-dimensional objects such as D-branes, NS-branes, and M-branes. All this has brought up new questions related to the flux and energy that each string type can carry to be stable, bringing in the grand scale of cosmological context the fascinating issue of the relations between topological lower bounds and admissible minimum energy.

Conclusion

The study of fluid flows and their continuous evolution provides the natural setting for the application and use of topological concepts and techniques. Topological fluid mechanics, and the study of knotted fields in particular, represent now a rich and diversified field of active research, where modern analytical techniques and advanced computational methods provide continuing news of exciting and fascinating ideas for future developments in science and technology. The symbiotic relationship between topology and fluid mechanics is in this context set to stay and grow for the forseeable future.

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