In *Poster Papers Presented at the VII European Meeting on Solar Physics*
Catania Astrophysical Obs. (ed. G. Belvedere *et al.*), pp. 151–154, 1994.

WRITHE AND TWIST HELICITY CONTRIBUTIONS TO A MAGNETIC FLUX LOOP AND HAMMOCK CONFIGURATION

RENZO L. RICCA

Mathematics Department, University College London, Gower Street, London WC1E 6BT, UK

1. THE HELICITY OF A MAGNETIC FLUX TUBE VIA THE CĂLUGĂREANU INVARIANT

It is known that the magnetic structures present in the solar atmosphere can be considered as embedded in a perfectly conducting fluid (at least over the characteristic length scales and time scales of motion), in which the topology of the field lines is preserved. Plasma loops in the solar corona can be conveniently modelled by magnetic flux tubes evolving in an ideal magnetohydrodynamical (MHD) context. For simplicity, let us assume that the coronal domain is filled by incompressible and inviscid fluid where no singularities are present (for example, no current sheets); then, the solenoidal magnetic field $\mathbf{B} = \mathbf{B}(\mathbf{X}, t) = \nabla \times \mathbf{A}(\mathbf{X}, t)$, function of the position vector $\mathbf{X}$ and time t, is advected by the fluid according to the Cauchy's equation

$$B_i(\mathbf{X}, t) = B_j(\mathbf{a}, 0)\frac{\partial X_i}{\partial a_j} \quad , \tag{1}$$

a result that incorporates both the convection of the magnetic field from the position $\mathbf{a}$ to the position $\mathbf{X}$ and its rotation and distortion by the deformation tensor $\partial X_i/\partial a_j$. The ambient fluid flow induces a map which is a continuous, time-dependent, orientable and volume-preserving diffeomorphism $\mathbf{a} \to \mathbf{X}$ of the fluid onto itself. Although the geometrical properties of the magnetic structures can become very complex in time, their topological properties (such as linking and knottedness) remain invariant during the motion.

Suppose now that $\mathbf{B}$ is zero except in a thin magnetic flux tube of volume $\mathcal{V}$ and circular cross-section. The magnetic helicity $\mathcal{H}$ of $\mathbf{B}$, defined by

$$\mathcal{H} = \int_{\mathcal{V}} \mathbf{A} \cdot \mathbf{B}\, dV \quad , \tag{2}$$

where $\mathbf{n} \cdot \mathbf{B} = 0$ on the boundary of $\mathcal{V}$ and dV is the volume element $d^3\mathbf{X}$, is known to be a conserved quantity under 'frozen field' distortion of the ambient medium (Woltjer, 1958) and admits a topological interpretation (Moreau, 1961; Moffatt, 1969) in terms of the Călugăreanu invariant $\mathcal{L}$, a measure of the self-linkage of a reference ribbon (Berger & Field, 1984; Ricca & Moffatt, 1992). To clarify this point we make use of a ribbon model for the magnetic flux tube: let $C : \mathbf{X} = \mathbf{X}(s)$ be the magnetic centreline of the flux tube, parametrised by arc length s, and let $\mathbf{X}^* = \mathbf{X} + \epsilon\mathbf{N}$ be a neighbouring magnetic field line running all along $\mathbf{X}$ and placed a small distance ϵ apart along a normal direction $\mathbf{N}$ to $\mathbf{X}$. Let $\mathbf{X}$ and $\mathbf{X}^*$ be the edges of a ribbon of spanwise width ϵ and let Θ be the angle of twist of the spanwise vector $\mathbf{N}$ relative to the Frenet vectors $(\mathbf{n}, \mathbf{b})$ (principal normal and binormal to $\mathbf{X}$). The (Călugăreanu) invariant $\mathcal{L}$ is the limiting form of the (Gauss) linking number of $\mathbf{X}$ with $\mathbf{X}^*$ as $\epsilon \to 0$. A direct derivation of this topological invariant from the invariance of helicity has been given by Moffatt & Ricca (1992), showing that

$$\mathcal{H} = \mathcal{L}\Phi^2 = (\mathcal{W} + Tw)\Phi^2 \quad , \tag{3}$$

where the *writhe* $\mathcal{W}$ is given by

$$\mathcal{W} = \frac{1}{4\pi}\oint_C \oint_C \frac{(d\mathbf{X} \times d\mathbf{X}^*) \cdot (\mathbf{X} - \mathbf{X}^*)}{|\mathbf{X} - \mathbf{X}^*|^3} \quad , \tag{4}$$

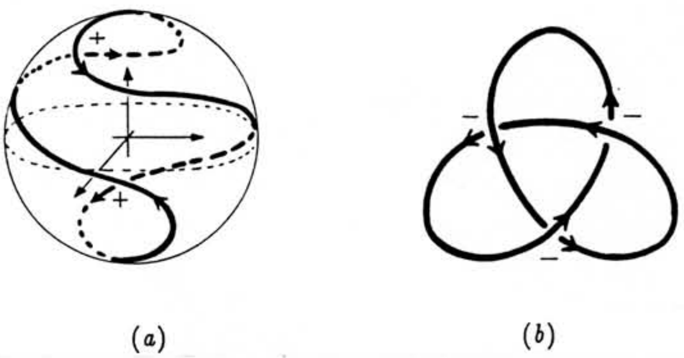


FIGURE 1. (a) The writhe of the plane projection of the space curve shown in the figure is +2, but its value avaraged over all projections is zero. (b) Nearly plane knotted curve (except indentations in the plane to allow crossings) with $\mathcal{W} = -3$.

and the *total twist* Tw is given by

$$Tw = \frac{1}{2\pi} \oint_C \left(\tau + \frac{d\Theta}{ds} \right) ds \quad , \tag{5}$$

the sum of the normalised integral of torsion (τ) and of the gradient of the twist angle ($d\Theta/ds$); Φ is the magnetic flux.

2. PHYSICAL INTERPRETATION OF WRITHE AND TWIST CONTRIBUTIONS TO HELICITY

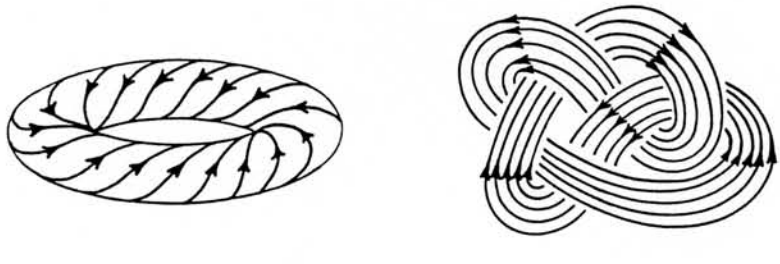
Equation (3) shows that $\mathcal{H}$ can be decomposed in two parts, the writhe and the twist contribution. Although the sum of these two quantities is a topological invariant (helicity does not change under continuous deformations), the writhe and twist contributions, taken separately, are not invariant and their values change according with the change of geometry. This means that $\mathcal{W}$ and Tw interchange continuously during the deformation.

The writhe $\mathcal{W}$, given by eq. (4), is carachterised by the following properties: i) $\mathcal{W}$ depends only on the geometry of the tube axis C; ii) $\mathcal{W}$ is invariant under rigid motions or dilations of the space containing C (conformal invariant), but its sign is changed by reflection; iii) in passing from an undercrossing to an overcrossing of the strands of C (in some projection plane), its value jumps by +2. The physical meaning of the writhe is evident by a close inspection of the integrand in eq. (4) and the definition of solid angle: it has been shown (Fuller, 1971; Moffatt & Ricca; 1992) that $\mathcal{W}$ can be interpreted in terms of the sum of signed crossings of the diagram of C in some projection plane, averaged over all projections, namely

$$\mathcal{W} = < n_+(\boldsymbol{\nu}) - n_-(\boldsymbol{\nu}) > \quad , \tag{6}$$

where the angular brackets denote averaging over all directions $\boldsymbol{\nu}$ of projection and $n_\pm$ denotes the number of ($\pm$) crossings apparent from the direction of projection $\boldsymbol{\nu}$ (figure 1a). For a nearly plane curve (except small indentations to allow crossings) the writhe can be directly estimated by the sum of the signed crossings (figure 1b).

The total twist Tw is given by eq. (5) and has the following properties: i) Tw is a continuous function of C (even through self-intersection); ii) Tw is invariant under rigid motions or dilations of the space containing the flux tube (conformal invariant), but its sign is changed by reflection; iii) Tw is additive: $Tw(A + B) = Tw(A) + Tw(B)$, where A and B are two contiguous sub-strips of the ribbon $A + B$. Part of the twist contribution to magnetic helicity is associated with the



(a) (b)

FIGURE 2. In (a) the intrinsic twist of the field lines contributes to helicity, whereas the writhe and the torsion contributions to helicity are identically zero. In (b) the writhe and the torsion contributions to helicity are decidedly *not* zero and there is no contribution from the internal twist of the field lines.

torsion of the tube axis C and part with what may be described as 'intrinsic twist' of the field lines in the flux tube around C (figure 2).

Note that the total twist is a continuous function of C, but total torsion and intrinsic twist are discontinuous in deformations that take the tube axis through an inflexion point (a point at which the curvature vanishes) (Ricca & Moffatt, 1992; Moffatt & Ricca, 1992). Inflexional configurations occur when twists are converted into writhes or vice versa. The occurrence of the mechanism that converts twist into writhe is at the origin of the kink instability and plays a part in the generation of a mechanical (and magnetic) support of cool plasma by critically twisted magnetic flux loops, as discussed in the following section.

3. HAMMOCK CONFIGURATION OF A MAGNETIC LOOP-LIKE STRUCTURE

It is generally accepted that the coronal motion of magnetic structures is governed by photospheric footpoint motions via line-tied field lines. The line-tying assumption provides a natural mechanism for the generation of internal field line twisting in a flux loop by vortical motion at the photospheric boundary. It has been shown (see, for example, Priest *et al.*, 1989; Finn *et al.*, 1993, submitted) that at a critical footpoint twist, individual field lines may acquire a concave-up curvature near their summits and, as a consequence, an initially simply bent magnetic flux loop may take a suitable shape (a hammock-shaped configuration) so as to provide support for cool plasma. In the force-free equilibrium, it has been found that a dip is produced at a critical twist of order 2π.

A simple kinematical model can be given based on the physical interpretation of writhe and twist helicity contributions. Consider a semi-circular magnetic flux tube (half a torus) with no internal twist of the field lines, whose footpoints are rooted in a vortical photospheric region (figure 3a). The initial (open-)helicity of this configuration is zero. As vortical and shearing boundary motions generate some twisting within the flux tube (as discussed, for example, by Berger, 1991) (figure 3b), twist helicity increases. This process stops only when a critical value for the internal twisting of the field lines is reached, the threshold being determined by the geometry of the flux tube and the magnetic tension.

When internal twist can no longer be introduced, any further increase in twist helicity by boundary motions is converted into writhe helicity until a force-free equilibrium is reached and, correspondingly, a dip is formed. After this stage, a further writhing of the flux tube is also impeded and a kink mode instability develops. In forming a hammock configuration, the flux tube centreline must pass through an inflexional configuration where curvature is zero, since a change

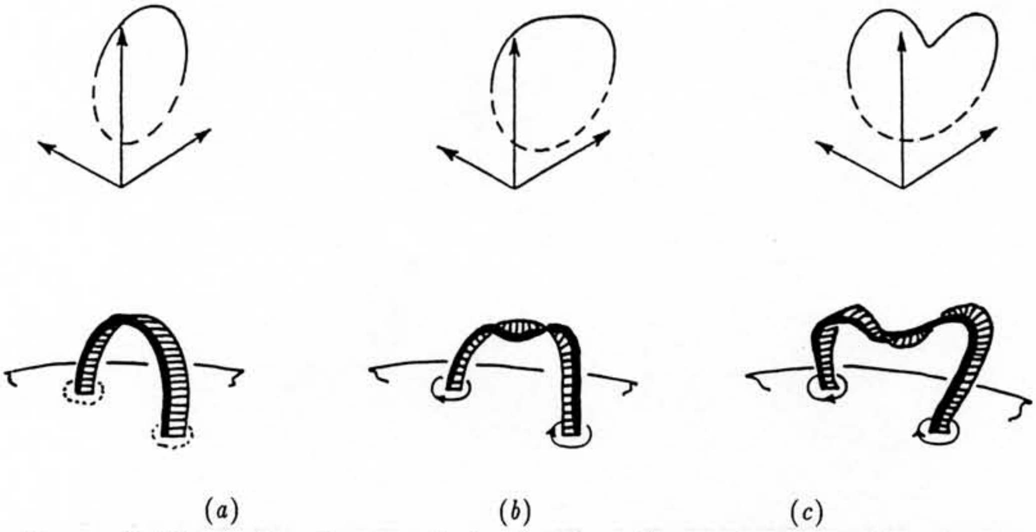


FIGURE 3. Deformation of a magnetic flux loop into a hammock configuration: above, 3D plots of the tube axis evolution as given by eqs. (7); below, internal twist pictured by a ribbon model. (*a*) Case of zero twist. (*b*) Case of half a twist. (*c*) Case of one footpoint revolution of the field lines, for which a hammock configuration is realised.

in concavity is invariably associated with the presence of inflexion points (figure 3*c*). As the twist increases, the tube axis $\mathbf{X} = (x, y, z)$ is generically deformed according to the equations

$$\left.\begin{aligned} x &= a\cos\xi - c(Tw)\cos 2\xi \\ y &= b\sin\xi - c(Tw)\sin 2\xi \\ z &= c(Tw)\sin\xi \end{aligned}\right\} \quad \begin{aligned} &\xi \in [-\frac{\pi}{2}, +\frac{\pi}{2}] \\ &Tw \in [0, 1] \end{aligned} \tag{7}$$

where a, b, c are configuration parameters (which depend on the MHD evolution), ξ is a polar coordinate and Tw is the total twist. Three-dimensional plots of $\mathbf{X}$ for $Tw = 0, \frac{1}{2}, 1$ (and $a = b = 1$, $c = 1/2$) are shown in figure 3. The kink mode instability that converts twist to writhe is naturally associated with a transition to a lower energy state (Ricca, 1993, in preparation). This analysis will be presented in a future paper.

ACKNOWLEDGMENTS. Financial support from UK Science and Engineering Research Council Grant No. GR/G46745 is kindly acknowledged.

REFERENCES

BERGER, M.A. 1991 *Astron. Astrophys.* **252**, 369–376.

BERGER, M.A. AND FIELD, G.B. 1984 *J. Fluid Mech.* **147**, 133–148.

FINN, J.M., GUZDAR, P.N. AND USIKOV, D. 1993 Submitted to *Astrophys. J.*.

FULLER, F.B. 1971 *Proc. Natl. Acad. Sci. USA* **68**, 815–819.

MOFFATT, H.K. 1969 *J. Fluid Mech.* **35**, 117–129.

MOFFATT, H.K. AND RICCA, R.L. 1992 *Proc. R. Soc. Lond.* A **439**, 411–429.

MOREAU, J.J. 1961 *C. R. Acad. Sci. Paris* **252**, 2810–2812.

PRIEST, E.R., HOOD, A.W. AND ANZER, U. 1989 *Astrophys. J.* **344**, 1010–1025.

RICCA, R.L. 1993. In preparation.

RICCA, R.L. AND MOFFATT, H.K. 1992 In *Topological aspects of the dynamics of fluids and plasmas* (ed. H.K. Moffatt *et al.*), pp. 225–236. Dordrecht, Kluwer.

WOLTJER, L. 1958 *Proc. Natl. Acad. Sci. USA* **44**, 489–491.