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Hydrodynamic derivation of the Gross–Pitaevskii equation in general Riemannian metric

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Abstract
Here we show that the Gross–Pitaevskii equation (GPE) for Bose–Einstein condensates (BECs) admits hydrodynamic interpretation in a general Riemannian metric, and show that in this metric the momentum equation has a new term that is associated with local curvature and density distribution profile. In particular conditions of steady state a new Einstein's field equation is determined in presence of negative curvature. Since GPE governs BECs defects that are useful, analogue models in cosmology, a relativistic form of GPE is also considered to show connection with models of analogue gravity, thus providing further grounds for future investigations of black hole dynamics in cosmology.

Keywords: Gross–Pitaevskii equation in Riemannian metric, GPE hydrodynamics, relativistic GPE

(Some figures may appear in colour only in the online journal)

1. Introduction

In this paper we derive the Euler and Navier–Stokes form of the Gross–Pitaevskii equation (GPE) [1, 2] in general Riemannian metric, and show that the metric gives rise to a new term in the corresponding momentum equation, which is associated with the geometry of density distribution profile. As a by-product of this new set of equations we derive in the stationary case a new Einstein's field equation. As we shall point out by considering the relativistic form of GPE in appropriate metric, these results provide further grounds for theoretical and experimental work [3] in connection with analogue models of gravity. Since GPE governs vortex

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defects in Bose–Einstein condensates (BECs) [4], the present work provides also a dynamical model for the study of defects in cosmological contexts, such as black hole theory.

From the first laboratory realization of BECs [5] and production of vortex defects [6] to the recent observational evidence of rotating black holes [7], the idea that massive gravitational vortex defects could develop from a condensed state of matter has become a subject of intense research [8–11]. The study of topological defects in cosmology developed more markedly in the 90s [12], where models of vortex string-like objects embedded in a higher-dimensional space were considered [13]. New impetus to the study of vortex defects came with the recent work on BECs, where superposition of twist effects on existing defects led to investigate new interesting relations between physical and topological properties of such systems [14], with novel implications for cosmological models of the Universe [15]. Analogue models of gravity [16] appealing to curved spacetime metrics have been receiving considerable attention since Unruh's [8] (and Visser's [17]) first proposal. Broadly speaking these models rely on the analogy between black hole thermal radiation and sound emission on an effective $(n + 1)$-dimensional spacetime geometry. In this context curved spacetime plays a crucial role, especially because sound waves in a transonic fluid flow are found to propagate along geodesics of an acoustic spacetime metric, providing grounds for the analogy between the behavior of light waves in black hole theory and acoustic waves in condensates.

These new developments led to a series of works devoted to prove analogue properties also between BECs hydrodynamics and gravity in a curved spacetime effective Lorentzian geometry (see, for example, [18–21]), where kinematic aspects associated with black hole formation and Hawking radiation have been explored [22]. Garay *et al* [19, 23], in particular, showed that in the GPE hydrodynamic limit, configurations that exhibit behaviors analogue to that of a sonic black hole do exist. Moreover, Visser *et al* proved [24] that the propagation of phononic perturbations in a fluid can be described by a wave equation defined in an effective relativistic curved spacetime metric entirely determined by the physical properties of the system, thus demonstrating that in the low momentum limit BECs may provide a useful model for an approximate Lorentz invariance. The fact that in the low momentum regime phase perturbations of the governing wave equation behave as if they were coupled to an effective Lorentzian metric is actually a general property of a wide class of non-linear Schrödinger equations, thus including the GPE, ensuring that such a metric is also a necessary pre-requisite for the existence of the Hawking radiation. Following [25], by identifying the missing dark energy with a BEC state, Fukuyama and Morikawa [26] considered a relativistic form of GPE in curved spacetime. This approach offers good account of various stages of cosmic evolution, from inflation and conversion of dark energy to dark matter, to the very early formation of black holes [27, 28], while providing further information for laboratory experiments. As we shall point out in the last section below, a relativistic form of GPE represents indeed an appropriate corollary to the rigorous results presented here, giving grounds to our extended hydrodynamic description of BECs in curved spacetime.

The material is presented as follows. In the next section we demonstrate that GPE admits hydrodynamic forms in any general Riemannian metric. In section 3 we show that GPE, standardly reduced to a continuity and a momentum equation, can be expressed in Euler form. In section 4 we show that in absence of a trapping potential (allowing free expansion of the boson gas) GPE can also be expressed in Navier–Stokes form, with the appearance of a new term that depends on the manifold geometry and density distribution profile. In section 5 we demonstrate that GPE in steady conditions, cast in the form of a conservation law for momentum, can be reduced to a type of Einstein's field equation. Finally, in section 6 we recall the relativistic form of GPE and show its reduction to the standard Euclidean form of GPE. Concluding remarks are presented in section 7, with emphasis on the role of curvature and density profile on dynamics.

2. Hydrodynamic form of the Gross–Pitaevskii equation on a generic Riemannian manifold

Let us consider the GPE [1, 2] in the standard form given by

$$\mathrm{i}\,\hbar\,\partial_t\Psi=\left(-\frac{\hbar^2}{2m}\,\boldsymbol{\nabla}^2+\mathfrak{g}|\Psi|^2+V\right)\Psi, \tag{1}$$

where $\mathrm{i}=\sqrt{-1}$, $\hbar=h/2\pi$ (h Planck's constant), m boson's mass, $\mathfrak{g}=(4\pi\hbar^2a_\mathrm{s})/m$ (a_s scattering length), $V=V(\boldsymbol{x},t)$ the external potential (function of the position vector $\boldsymbol{x}$ and time t), and $\Psi=\Psi(\boldsymbol{x},t)$ the complex wave function. In order to derive the hydrodynamic form of (1) on a generic Riemannian manifold M of metric $g_{ij}=g(\partial_i,\partial_j)$ we follow a standard procedure as in the Euclidean case [29]. First, let's recall the expression for the Laplace operator of Ψ, defined by

$$\boldsymbol{\nabla}^2\Psi=\frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}\,g^{ij}\partial_j\Psi\right), \tag{2}$$

where g^{ij} denotes the metric inverse matrix and $|g|=|\det(g_{ij})|$. By applying the Madelung transformation [30] $\Psi=\sqrt{\rho}\,\exp\left[\mathrm{i}(m/\hbar)\theta\right]$ function of density ρ and phase θ, the left-hand side of (1) becomes

$$\mathrm{i}\hbar\partial_t\Psi=\frac{\mathrm{i}\hbar}{2\sqrt{\rho}}(\partial_t\rho)\,\mathrm{e}^{\mathrm{i}\left(\frac{m}{\hbar}\theta\right)}-m\sqrt{\rho}\,\mathrm{e}^{\mathrm{i}\left(\frac{m}{\hbar}\theta\right)}\partial_t\theta, \tag{3}$$

the Laplacian term becomes

$$\begin{aligned}\boldsymbol{\nabla}^2\Psi&=\frac{1}{\sqrt{|g|}}\partial_i\left[\sqrt{|g|}\,g^{ij}\partial_j\left(\sqrt{\rho}\,\mathrm{e}^{\mathrm{i}\left(\frac{m}{\hbar}\theta\right)}\right)\right]\\&=\left[\boldsymbol{\nabla}^2\sqrt{\rho}+\mathrm{i}\frac{m}{\hbar}\left(\frac{1}{\sqrt{\rho}}\boldsymbol{\nabla}\rho\cdot\boldsymbol{\nabla}\theta+\sqrt{\rho}\,\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\theta)\right)-\frac{m^2}{\hbar^2}\sqrt{\rho}\,|\boldsymbol{\nabla}\theta|^2\right]\mathrm{e}^{\mathrm{i}\left(\frac{m}{\hbar}\theta\right)},\end{aligned} \tag{4}$$

and the other two terms on the right-hand side of (1) become

$$(\mathfrak{g}|\Psi|^2+V)\Psi=(\mathfrak{g}\rho+V)\sqrt{\rho}\,\mathrm{e}^{\mathrm{i}\left(\frac{m}{\hbar}\theta\right)}. \tag{5}$$

By equating the imaginary parts we obtain an equation for the density that can be recast in the form of a continuity equation:

$$\partial_t\rho=-\boldsymbol{\nabla}\rho\cdot\boldsymbol{\nabla}\theta-\rho\,\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\theta)=-\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\nabla}\theta). \tag{6}$$

By equating the real parts we obtain a Bernoulli type equation for the phase θ, given by

$$\partial_t\theta+\frac{1}{2}|\boldsymbol{\nabla}\theta|^2=U+Q, \tag{7}$$

where $U=-(\mathfrak{g}\rho+V)/m$ is classical potential and $Q=(\hbar^2/2m^2)(\boldsymbol{\nabla}^2\sqrt{\rho})/\sqrt{\rho}$ is quantum potential.

3. GPE in Euler form

By defining the velocity field $\boldsymbol{u} = \boldsymbol{\nabla}\theta$, equation (6) takes the standard form of a continuity equation for the density ρ, given by

$$\partial_t \rho = -\boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}), \tag{8}$$

while for equation (7), we have:

Theorem 1 (GPE in Euler form). *In general Riemannian metric the GPE admits hydrodynamic description in the form of an Euler equation, given by*

$$\partial_t \boldsymbol{u} + \boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{u} = \boldsymbol{\nabla} H, \tag{9}$$

where $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{u}$ denotes the covariant derivative of the velocity field $\boldsymbol{u}$ along itself, and $H = U + Q$ is the sum of the classical potential U and the quantum potential Q.

Proof. First, let's re-write the covariant derivative using Koszul's formula [31, 32]

$$g(\boldsymbol{\nabla}_X Y, Z) = \frac{1}{2}\left[Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \right.$$
$$\left. - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y])\right]. \tag{10}$$

Let $Y = X$; we have

$$g(\boldsymbol{\nabla}_X X, Z) = \frac{1}{2}\left[2Xg(X,Z) - Z(|X|^2) + 2g(X,[Z,X])\right], \tag{11}$$

and by taking $X = \boldsymbol{\nabla}\theta$, we obtain

$$g(\boldsymbol{\nabla}_{\boldsymbol{\nabla}\theta}\boldsymbol{\nabla}\theta, Z) = \boldsymbol{\nabla}\theta\,(g(\boldsymbol{\nabla}\theta, Z)) - \frac{1}{2}Z(|\boldsymbol{\nabla}\theta|^2) + g(\boldsymbol{\nabla}\theta,[Z,\boldsymbol{\nabla}\theta])$$
$$= \boldsymbol{\nabla}\theta\,(\mathrm{d}\theta(Z)) - \frac{1}{2}Z(|\boldsymbol{\nabla}\theta|^2) + \mathrm{d}\theta([Z,\boldsymbol{\nabla}\theta]). \tag{12}$$

Since

$$\mathrm{d}\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y]), \tag{13}$$

by taking $X = Z$, $Y = \boldsymbol{\nabla}\theta$ and $\alpha = \mathrm{d}\theta$, we have

$$0 = Z\,(\mathrm{d}\theta(\boldsymbol{\nabla}\theta)) - \boldsymbol{\nabla}\theta\,(\mathrm{d}\theta(Z)) - \mathrm{d}\theta\,([Z,\boldsymbol{\nabla}\theta]), \tag{14}$$

so that

$$\boldsymbol{\nabla}\theta\,(\mathrm{d}\theta(Z)) + \mathrm{d}\theta\,([Z,\boldsymbol{\nabla}\theta]) = Z(\mathrm{d}\theta(\boldsymbol{\nabla}\theta)) = Z\,(g(\boldsymbol{\nabla}\theta,\boldsymbol{\nabla}\theta)) = Z(|\boldsymbol{\nabla}\theta|^2). \tag{15}$$

Using (12) and (15), we thus have

$$g(\boldsymbol{\nabla}_{\boldsymbol{\nabla}\theta}\boldsymbol{\nabla}\theta, Z) = \frac{1}{2}Z(|\boldsymbol{\nabla}\theta|^2). \tag{16}$$

In local coordinates the term above can be written as

$$\frac{1}{2}Z(|\boldsymbol{\nabla}\theta|^2)(x) = \frac{1}{2}\mathrm{d}_x(|\boldsymbol{\nabla}\theta|^2)Z(x), \tag{17}$$

where $x \in M$; we also have

$$g\left[\boldsymbol{\nabla}\left(\frac{1}{2}|\boldsymbol{\nabla}\theta|^2\right), Z\right](x) = \mathrm{d}_x\left(\frac{1}{2}|\boldsymbol{\nabla}\theta|^2\right) Z(x). \tag{18}$$

Hence, from (16) and (18) we have

$$g(\boldsymbol{\nabla}_{\boldsymbol{\nabla}\theta}\boldsymbol{\nabla}\theta, Z) = g\left[\boldsymbol{\nabla}\left(\frac{1}{2}|\boldsymbol{\nabla}\theta|^2\right), Z\right], \tag{19}$$

valid for all vector fields Z; thus, we have

$$\boldsymbol{\nabla}_{\boldsymbol{\nabla}\theta}\boldsymbol{\nabla}\theta = \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{u} = \boldsymbol{\nabla}\left(\frac{1}{2}|\boldsymbol{\nabla}\theta|^2\right). \tag{20}$$

Taking the gradient of (7) and using (20), we have (9), proving that the relation between GPE and hydrodynamics is one to one on any manifold M with generic metric g. □

4. GPE in Navier–Stokes form

In order to consider the free expansion of the gas from its original state, it is worth exploring the case of an unconstrained state of matter in absence of trapping potential. In this case we can prove the following result:

Theorem 2 (GPE in Navier–Stokes form). *In general Riemannian metric the GPE admits hydrodynamic description in the form of a Navier–Stokes equation, given by*

$$\rho\,(\partial_t\boldsymbol{u} + \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{u}) = -\boldsymbol{\nabla}\wp + \boldsymbol{\nabla}\cdot\boldsymbol{\tau} + \boldsymbol{\mathcal{E}}, \tag{21}$$

where for simplicity (and without loss of generality) we set $V = 0$, *and where* $\wp = (\mathfrak{g}/2m)\rho^2$ *is a pressure-like term,* $\boldsymbol{\tau} = (\hbar^2/4m^2)\rho\,\mathrm{Hess}(\ln\rho)$ *is a stress-tensor term, and* $\boldsymbol{\mathcal{E}}$ *is a density curvature vector given by* $\mathcal{E}^j = -(\hbar^2/4m^2)R^{jk}\partial_k\rho$, *with* R^{jk} *Ricci's tensor.*

Proof. Let us re-write (9), multiplying everything by ρ and setting $V = 0$; we have

$$\rho(\partial_t\boldsymbol{u} + \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{u}) = -\frac{\mathfrak{g}}{m}\rho\boldsymbol{\nabla}\rho + \frac{\hbar^2}{m^2}\rho\boldsymbol{\nabla}\left(\frac{\boldsymbol{\nabla}^2\sqrt{\rho}}{2\sqrt{\rho}}\right). \tag{22}$$

Evidently, by taking $\wp = (\mathfrak{g}/2m)\rho^2$ the first term on the right-hand side of the equation above can be written as $-(\mathfrak{g}/m)\rho\boldsymbol{\nabla}\rho = -\boldsymbol{\nabla}\wp$. Then, notice that

$$\frac{\boldsymbol{\nabla}^2\sqrt{\rho}}{2\sqrt{\rho}} = \frac{1}{2\sqrt{\rho}}\frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j\sqrt{\rho}\right) = \frac{1}{4\rho}\boldsymbol{\nabla}^2\rho - \frac{1}{8\rho^2}|\boldsymbol{\nabla}\rho|^2, \tag{23}$$

so that the second term on the right-hand side of (22), written as one-form, becomes

$$\frac{\hbar^2}{m^2}\rho\,\mathrm{d}\left(\frac{\boldsymbol{\nabla}^2\sqrt{\rho}}{2\sqrt{\rho}}\right) = \frac{\hbar^2}{4m^2}\rho\,\mathrm{d}\left(\frac{1}{\rho}\boldsymbol{\nabla}^2\rho - \frac{1}{2\rho^2}|\boldsymbol{\nabla}\rho|^2\right). \tag{24}$$

Now, let's show that

$$\rho\,\mathrm{d}\left(\frac{1}{\rho}\boldsymbol{\nabla}^2\rho - \frac{1}{2\rho^2}|\boldsymbol{\nabla}\rho|^2\right) = \boldsymbol{\nabla}\cdot(\rho\,\mathrm{Hess}(\ln\,\rho)) + \tilde{\boldsymbol{\mathcal{E}}}^\flat, \tag{25}$$

where $\tilde{\boldsymbol{\mathcal{E}}}^\flat = (4m^2/\hbar^2)\mathcal{E}_j\,\mathrm{d}x^j$. Let's re-write the left-hand side of (25) in terms of coordinates; we have

$$\begin{aligned}\rho\,\mathrm{d}\left(\frac{1}{\rho}\boldsymbol{\nabla}^2\rho-\frac{1}{2\rho^2}|\boldsymbol{\nabla}\rho|^2\right) &= \rho\,\mathrm{d}\left(\frac{1}{\rho}g^{ik}\partial^2_{ik}\rho-\frac{1}{\rho}g^{ik}\Gamma^r_{ik}\partial_r\rho\right.\\ &\quad\left.-\frac{1}{2\rho^2}g^{ik}(\partial_i\rho)\partial_k\rho\right)\\ &= \left[g^{ik}\left(-\frac{1}{\rho}(\partial_j\rho)\partial^2_{ik}\rho+\partial^3_{jik}\rho+\frac{1}{\rho}\Gamma^r_{ik}(\partial_r\rho)\partial_j\rho\right.\right.\\ &\quad-(\partial_j\Gamma^r_{ik})\partial_r\rho-\Gamma^r_{ik}\partial^2_{rj}\rho+\frac{1}{\rho^2}(\partial_j\rho)(\partial_i\rho)\partial_k\rho\\ &\quad\left.-\frac{1}{\rho}(\partial_i\rho)\partial^2_{jk}\rho\right)+(\partial_j g^{ik})\,(\partial^2_{ik}\rho-\Gamma^r_{ik}\partial_r\rho\\ &\quad\left.-\frac{1}{2\rho}(\partial_i\rho)\partial_k\rho)\right]\mathrm{d}x^j, \end{aligned} \tag{26}$$

where Γ^i_{jk} denotes Christoffell's symbol. Further manipulations on the metric gives

$$\begin{aligned}\partial_j g^{ik} &= -g^{ir}(\partial_j g_{rs})g^{sk} = -g^{ir}\left(g_{as}\Gamma^a_{jr}+g_{rb}\Gamma^b_{js}\right)g^{sk}\\ &= -g^{ir}\Gamma^k_{jr}-g^{sk}\Gamma^i_{js},\end{aligned} \tag{27}$$

and because the sum is carried over i and k, from the last term we have $\partial_j g^{ik} = -2g^{is}\Gamma^k_{js}$; thus, equation (26) becomes

$$\begin{aligned}\rho\,\mathrm{d}\left(\frac{1}{\rho}\boldsymbol{\nabla}^2\rho-\frac{1}{2\rho^2}|\boldsymbol{\nabla}\rho|^2\right) &= \left[g^{ik}\left(-\frac{1}{\rho}(\partial_j\rho)\partial^2_{ik}\rho+\partial^3_{jik}\rho+\frac{1}{\rho}\Gamma^r_{ik}(\partial_r\rho)\partial_j\rho\right.\right.\\ &\quad-(\partial_j\Gamma^r_{ik})\partial_r\rho-\Gamma^r_{ik}\partial^2_{rj}\rho+\frac{1}{\rho^2}(\partial_j\rho)(\partial_i\rho)\partial_k\rho\\ &\quad\left.-\frac{1}{\rho}(\partial_i\rho)\partial^2_{jk}\rho\right)-2g^{is}\Gamma^k_{js}\partial^2_{ik}\rho\\ &\quad\left.+2g^{is}\Gamma^k_{js}\Gamma^r_{ik}\partial_r\rho+\frac{1}{\rho}g^{is}\Gamma^k_{js}(\partial_i\rho)\partial_k\rho\right]\mathrm{d}x^j. \end{aligned} \tag{28}$$

Now let's consider the right-hand side of (25); from the definition of Hessian, we have

$$\begin{aligned}\rho\,\mathrm{Hess}(\ln\,\rho) &= \rho\left(\partial^2_{ij}\,\ln\,\rho-\Gamma^r_{ij}\partial_r\,\ln\,\rho\right)\mathrm{d}x^i\wedge\mathrm{d}x^j\\ &= \rho\,\left[\partial_i\left(\frac{1}{\rho}\partial_j\rho\right)-\Gamma^r_{ij}\frac{1}{\rho}\partial_r\rho\right]\mathrm{d}x^i\wedge\mathrm{d}x^j\\ &= \left(-\frac{1}{\rho}(\partial_i\rho)\partial_j\rho+\partial^2_{ij}\rho-\Gamma^r_{ij}\partial_r\rho\right)\mathrm{d}x^i\wedge\mathrm{d}x^j\\ &= H_{ij}\mathrm{d}x^i\wedge\mathrm{d}x^j = \boldsymbol{\mathcal{H}}.\end{aligned} \tag{29}$$

The divergence of $\boldsymbol{\mathcal{H}}$ is defined by

$$\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{H}} = g^{ik}(\partial_k H_{ij}-H_{lj}\Gamma^l_{ki}-H_{il}\Gamma^l_{kj})\mathrm{d}x^j, \tag{30}$$

so that

$$\nabla \cdot \boldsymbol{\mathcal{H}} = g^{ik}\left[\partial_k\left(-\frac{1}{\rho}(\partial_i\rho)\partial_j\rho + \partial^2_{ij}\rho - \Gamma^r_{ij}\partial_r\rho\right)\right.$$
$$- \Gamma^l_{ki}\left(-\frac{1}{\rho}(\partial_l\rho)\partial_j\rho + \partial^2_{lj}\rho - \Gamma^r_{lj}\partial_r\rho\right)$$
$$\left.- \Gamma^l_{kj}\left(-\frac{1}{\rho}(\partial_l\rho)\partial_i\rho + \partial^2_{li}\rho - \Gamma^r_{li}\partial_r\rho\right)\right]\mathrm{d}x^j; \tag{31}$$

hence, after some algebra, we have

$$\nabla \cdot \boldsymbol{\mathcal{H}} = g^{ik}\left[\frac{1}{\rho^2}(\partial_k\rho)(\partial_i\rho)\partial_j\rho - \frac{1}{\rho}(\partial^2_{ki}\rho)\partial_j\rho\right.$$
$$- \frac{1}{\rho}(\partial_i\rho)\partial^2_{kj}\rho + \partial^3_{kij}\rho - (\partial_k\Gamma^r_{ij})\partial_r\rho - \Gamma^r_{ij}\partial^2_{kr}\rho$$
$$+ \frac{1}{\rho}\Gamma^l_{ki}(\partial_l\rho)\partial_j\rho - \Gamma^l_{ki}\partial^2_{lj}\rho + \Gamma^l_{ki}\Gamma^r_{lj}\partial_r\rho$$
$$\left.+ \frac{1}{\rho}\Gamma^l_{kj}(\partial_l\rho)\partial_i\rho - \Gamma^l_{kj}\partial^2_{li}\rho + \Gamma^l_{kj}\Gamma^r_{li}\partial_r\rho\right]\mathrm{d}x^j. \tag{32}$$

After substituting (28) and (32) into (25) we can check that what is left out is $\tilde{\boldsymbol{\mathcal{E}}}^\flat = \tilde{\mathcal{E}}_j\mathrm{d}x^j$. After straightforward simplifications and index re-labeling, we obtain

$$\tilde{\mathcal{E}}_j = g^{ik}\left(\partial_k\Gamma^r_{ij} - \partial_j\Gamma^r_{ik} + \Gamma^l_{jk}\Gamma^r_{il} - \Gamma^l_{ki}\Gamma^r_{lj}\right)\partial_r\rho = g^{ik}R^r_{ikj}\partial_r\rho, \tag{33}$$

where R^r_{ikj} denotes Riemann's tensor. In terms of Ricci's curvature tensor R_{ij}, we have

$$\tilde{\mathcal{E}}_j = g^{ik}R^r_{ikj}\partial_r\rho = g^{ik}g^{lr}R_{likj}\partial_r\rho$$
$$= -g^{lr}R_{lj}\partial_r\rho = -g^{ik}R_{ij}\partial_k\rho. \tag{34}$$

Writing this term in vector form, we have

$$\tilde{\mathcal{E}}^l = g^{lj}\tilde{\mathcal{E}}_j = -g^{lj}g^{ik}R_{ij}\partial_k\rho = -R^{lk}\partial_k\rho. \tag{35}$$

Combining (24) and (25), and taking

$$\wp = \frac{\mathfrak{g}\rho^2}{2m}, \qquad \boldsymbol{\tau} = \frac{\hbar^2}{4m^2}\rho\,\mathrm{Hess}(\ln\ \rho), \qquad \boldsymbol{\mathcal{E}} = \frac{\hbar^2}{4m^2}\tilde{\boldsymbol{\mathcal{E}}},$$

we obtain equation (22). □

5. Conservation law for momentum

Using the continuity equation (8) and the Navier–Stokes form of GPE (9) we can prove a conservation law for the momentum.

Theorem 3 (Momentum conservation law). *The momentum $\rho\boldsymbol{u}$ associated with the hydrodynamic form of GPE satisfies the following conservation law*

$$\partial_t(\rho\boldsymbol{u}) = -\nabla \cdot \boldsymbol{\mathcal{M}} + \chi, \tag{36}$$

where $\boldsymbol{\mathcal{M}} = \mathcal{M}_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}x^j$ *is a tensor associated with the flux of momentum, given by*

$$\mathcal{M}_{ij} = \mathcal{D}_{ij} + \Pi_{ij} - \tau_{ij} - \mathcal{G}_{ij}, \tag{37}$$

with $\mathcal{D}_{ij} = \rho u_i u_j$, $\boldsymbol{u}^\flat = u_j \mathrm{d}x^j$, $\Pi_{ij} = \wp g_{ij}$, $\mathcal{G}_{ij}$ *an element of* $\boldsymbol{\mathcal{G}} = \hbar^2\rho/(4m^2)\boldsymbol{G}$ *(*$\boldsymbol{G}$ *Einstein's tensor), and* $\chi = -\hbar^2/(8m^2)R\,\mathrm{d}\rho$ *a term related to the geometry of M through the Ricci scalar curvature R.*

Proof. Let us consider $\partial_t(\rho\boldsymbol{u}) = (\partial_t\rho)\boldsymbol{u} + \rho\partial_t\boldsymbol{u}$: substitute equations (8) and (22) into this expression and write everything as one-form; we have

$$\partial_t(\rho\, u_j)\mathrm{d}x^j = A - \partial_j\wp\,\mathrm{d}x^j + g^{ik}\boldsymbol{\nabla}_k\tau_{ij}\,\mathrm{d}x^j - \frac{\hbar^2}{4m^2}g^{ik}R_{ij}\partial_k\rho\,\mathrm{d}x^j, \tag{38}$$

where, using the continuity equation (8), we set

$$\begin{aligned} A &= (\partial_t\rho)\boldsymbol{u} - \rho\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{u} = -\rho(\boldsymbol{\nabla}\cdot\boldsymbol{u})\boldsymbol{u} - (\boldsymbol{\nabla}\rho\cdot\boldsymbol{u})\boldsymbol{u} - \rho\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{u} \\ &= -\rho(\partial_k u^k + u^k\Gamma^l_{lk})u_j\,\mathrm{d}x^j - (\partial_k\rho)u^k u_j\,\mathrm{d}x^j - \rho(u^k\partial_k u_j - u^k u_l\Gamma^l_{kj})\mathrm{d}x^j. \end{aligned} \tag{39}$$

By writing $u^k = g^{ik}u_i$, from the equation above we have

$$\begin{aligned} A &= -\left[\rho\,(\partial_k u^k)u_j + \rho\,u^k(\partial_k u_j) + (\partial_k\rho)u^k u_j + \rho u^k\Gamma^l_{lk}u_j - \rho u^k u_l\Gamma^l_{kj}\right]\mathrm{d}x^j \\ &= -\left[\rho(\partial_k g^{ik})u_i u_j + g^{ik}\left(\partial_k(\rho u_i u_j) + \rho\,u_i u_j\Gamma^l_{lk} - \rho u_i u_l\Gamma^l_{kj}\right)\right]\mathrm{d}x^j. \end{aligned} \tag{40}$$

Now

$$\partial_k g^{ik} = -g^{ir}\Gamma^k_{kr} - g^{kr}\Gamma^i_{rk} = -g^{ik}\Gamma^l_{lk} - g^{kr}\Gamma^i_{rk}, \tag{41}$$

hence, by substituting the latter into the right-hand side of (40), after some index re-labeling we have that

$$A = -g^{ik}\left[\partial_k(\rho u_i u_j) - \rho u_i u_l\Gamma^l_{kj} - \rho u_l u_j\Gamma^l_{ik}\right]\mathrm{d}x^j. \tag{42}$$

Defining the $(0, 2)$-tensor $\boldsymbol{\mathcal{D}} = \mathcal{D}_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}x^j$, where $\mathcal{D}_{ij} = \rho u_i u_j$, the expression (39) reduces to

$$A = -g^{ik}\left(\partial_k\mathcal{D}_{ij} - \mathcal{D}_{il}\Gamma^l_{kj} - \mathcal{D}_{lj}\Gamma^l_{ki}\right)\mathrm{d}x^j = -g^{ik}\boldsymbol{\nabla}_k\mathcal{D}_{ij}\,\mathrm{d}x^j = -\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{D}}. \tag{43}$$

As regards the second term on the right-hand side of equation (38), we define the $(0, 2)$-tensor $\boldsymbol{\Pi} = \wp\, g_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j$ so to have

$$\boldsymbol{\nabla}\cdot\boldsymbol{\Pi} = g^{ik}\left(\partial_k(\wp\, g_{ij}) - \wp g_{il}\Gamma^l_{jk} - \wp g_{lj}\Gamma^l_{ik}\right)\mathrm{d}x^j = \partial_j\wp\,\mathrm{d}x^j, \tag{44}$$

where we used $\partial_k g_{ij} = g_{jl}\Gamma^l_{ik} + g_{il}\Gamma^l_{jk}$. Finally, as regards the last term on the right-hand side of equation (38) we want to express the density curvature tensor term $\tilde{\mathcal{E}}_j = -g^{ik}R_{ij}\partial_k\rho$ as divergence of a $(0, 2)$-tensor. Let's consider the term $\rho\boldsymbol{\mathcal{R}}$, where $\boldsymbol{\mathcal{R}}$ denotes Ricci's tensor. We have

$$\begin{aligned} \boldsymbol{\nabla}\cdot(\rho\boldsymbol{\mathcal{R}}) &= g^{ik}(\partial_k\rho)R_{ij}\,\mathrm{d}x^j + \rho g^{ik}\boldsymbol{\nabla}_k(R_{ij})\mathrm{d}x^j = -\tilde{\mathcal{E}}_j\,\mathrm{d}x^j + \rho\,\boldsymbol{\nabla}_k(g^{ik}R_{ij})\mathrm{d}x^j \\ &= -\tilde{\mathcal{E}}_j\,\mathrm{d}x^j + \rho\boldsymbol{\nabla}_k(R^k_j)\mathrm{d}x^j, \end{aligned} \tag{45}$$

where we used metric compatibility ($\nabla g = 0$). Hence, by the contracted form of Bianchi's identity $\nabla_k R^k_j \,\mathrm{d}x^j = (1/2)\partial_j R \,\mathrm{d}x^j$ (where R denotes Ricci's scalar curvature), we have

$$\tilde{\mathcal{E}}_j \,\mathrm{d}x^j = -\nabla \cdot (\rho \boldsymbol{\mathcal{R}}) + \frac{1}{2}\rho \partial_j R \,\mathrm{d}x^j. \tag{46}$$

Substituting (43), (44) and (46) into equation (38), we have

$$\partial_t(\rho\, u_j)\mathrm{d}x^j = -\nabla \cdot \left(D_{ij} + \wp\, g_{ij} - \tau_{ij} + \frac{\hbar^2}{4m^2}\rho R_{ij} \right) + \frac{\hbar^2}{8m^2}\rho \partial_j R \,\mathrm{d}x^j, \tag{47}$$

or

$$\partial_t(\rho \boldsymbol{u}) = -\nabla \cdot \left(\boldsymbol{\mathcal{D}} + \boldsymbol{\Pi} - \boldsymbol{\tau} + \frac{\hbar^2}{4m^2}\rho\, \boldsymbol{\mathcal{R}} \right) + \frac{\hbar^2}{8m^2}\rho \,\mathrm{d}R. \tag{48}$$

The equation above can be expressed in terms of Einstein's tensor $\boldsymbol{G}$, given by $G_{ij} = R_{ij} - R\, g_{ij}/2$, considering $\rho \boldsymbol{G} = G_{ij}\rho \ \mathrm{d}x^i \wedge \mathrm{d}x^j$; we have

$$\begin{aligned} \nabla \cdot (\rho \boldsymbol{G}) &= g^{ik}\nabla_k(\rho G_{ij})\mathrm{d}x^j = \left(g^{ik}(\partial_k \rho)G_{ij} + \rho g^{ik}\nabla_k G_{ij} \right) \mathrm{d}x^j \\ &= \left(g^{ik}(\partial_k \rho)R_{ij} - \frac{1}{2}g^{ik}(\partial_k \rho)g_{ij}R + \rho \nabla_k G^k_j \right) \mathrm{d}x^j \\ &= g^{ik}(\partial_k \rho)R_{ij} \,\mathrm{d}x^j - \frac{1}{2}R(\partial_j \rho)\mathrm{d}x^j, \end{aligned} \tag{49}$$

where we have used the fact that $\nabla_k G^k_j = 0$. Hence

$$\tilde{\mathcal{E}}_j \,\mathrm{d}x^j = -\nabla \cdot (\rho \boldsymbol{G}) - \frac{1}{2}R(\partial_j \rho)\mathrm{d}x^j. \tag{50}$$

Again, substituting (43), (44) and (50) into equation (38), we have

$$\partial_t(\rho \boldsymbol{u}) = -\nabla \cdot (\boldsymbol{\mathcal{D}} + \boldsymbol{\Pi} - \boldsymbol{\tau} + \boldsymbol{\mathcal{G}}) + \chi, \tag{51}$$

where $\boldsymbol{\mathcal{G}} = (\hbar^2/4m^2)\rho \boldsymbol{G}$, and

$$\chi = -\frac{\hbar^2}{8m^2}R \,\mathrm{d}\rho, \tag{52}$$

as stated by theorem 3. □

The new term χ breaks the standard form of a conservation law. As a point of digression it is perhaps worth noticing that there is a formal analogy with the right-hand side term that appears in the energy–momentum conservation law in general relativity [33, see section 126, equation (143)]. As pointed out by Penrose [34], though, since gravitational energy contributes non-locally to the total energy of the system and does not play an integral part in the energy and momentum conservation law associated with the Einstein field equation, a strict parallel with a standard conservation law does not hold in general relativity. In any case, the possibility of identifying a more stringent similarity between χ and the term mentioned above by writing χ in terms of divergence of a pseudo-tensor is certainly intriguing. As discussed in subsection 5.2 here below, this situation can arise in the case of constant Ricci curvature, where χ can indeed be absorbed into a divergence operator.

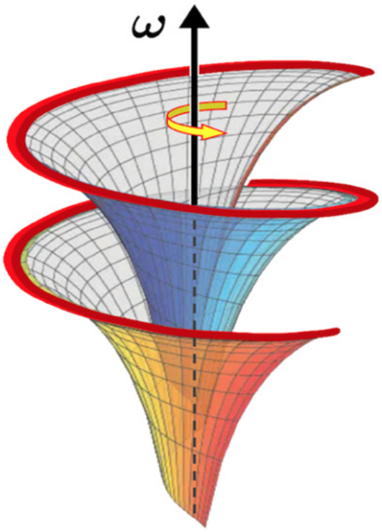

Figure 1. An example of possible black hole geometry given by the twisted elliptic pseudo-sphere (Dini's surface) associated with the presence of a central vortex defect represented by $\boldsymbol{\omega}$.

5.1. Stationary case

In general a steady state form of equation (36) is given by

$$\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{M}} = \chi. \tag{53}$$

A stationary state is also achieved in the very special situation when the two conditions

$$\text{(i)}\quad \boldsymbol{\mathcal{D}} - \boldsymbol{\tau} + \boldsymbol{\mathcal{G}} = 0\ , \qquad \text{(ii)}\quad \chi - \boldsymbol{\nabla}\cdot\boldsymbol{\Pi} = 0, \tag{54}$$

are simultaneously satisfied. From condition (i) we derive a new type of Einstein field equation, given by

$$G_{ij} = -\frac{4m^2}{\hbar^2}u_i u_j + \mathrm{Hess}_{ij}(\ln\ \rho). \tag{55}$$

This equation (mentioned without derivation) has been recently proposed [35] as a possible candidate for the appearance of cosmological singularities in black hole theory. From condition (ii) we determine the Ricci scalar curvature

$$R = -\frac{8m\mathfrak{g}}{\hbar^2}\rho = -32\pi a_{\mathrm{s}}\rho, \tag{56}$$

which implies a negative Gaussian curvature, given by $K = R/2 = -16\pi a_{\mathrm{s}}\rho$. In terms of healing length $\xi = (4\pi a_{\mathrm{s}}\rho)^{-1/2}$ [29] the Ricci scalar curvature takes the simple form $R = -8/\xi^2$, with Gaussian curvature $K = -4/\xi^2$. An interesting example of such a case is represented by the twisted pseudo-sphere shown in figure 1.

5.2. Constant curvature case

Evidently, in case of constant curvature we can re-write (52) as

$$-\frac{\hbar^2}{8m^2} R\,\mathrm{d}\rho = -\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{X}}, \tag{57}$$

where $\boldsymbol{\mathcal{X}} = \mathcal{X}_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}x^j$ with $\mathcal{X}_{ij} = \hbar^2/(8m^2)g_{ij}R\rho$, so that equation (36) can be written in the form of standard conservation law for the momentum

$$\partial_t(\rho\boldsymbol{u}) = -\boldsymbol{\nabla}\cdot(\boldsymbol{\mathcal{M}} + \boldsymbol{\mathcal{X}}), \tag{58}$$

obtaining again a steady state when $\boldsymbol{\mathcal{M}} + \boldsymbol{\mathcal{X}} = 0$, with Ricci tensor given by

$$R_{ij} = -\frac{4m^2}{\hbar^2}u_i u_j - 8\pi a_s g_{ij}\rho + \mathrm{Hess}_{ij}(\ln\,\rho). \tag{59}$$

6. Relativistic forms of GPE and hydrodynamics

It is worth mentioning the connection of the present derivation with current work on the relativistic forms of GPE when this is applied to cosmology. Following Fukuyama and Morikawa [26] and Matos and Gomez [36] let us consider the Lagrangian

$$L = g^{\mu\nu}(\partial_\nu\phi^*)(\partial_\nu\phi) - \frac{m^2c^2}{\hbar^2}\phi^*\phi - \frac{\lambda}{2\hbar^2}(\phi^*\phi)^2,$$

where $g^{\mu\nu}$ is the spacetime metric, ϕ (ϕ^* complex conjugate) the BEC's wave function, c the speed of light, and λ the coupling constant (positive or negative). By applying the Euler–Lagrange equation, we obtain

$$\partial_\mu\partial^\mu\phi = -\frac{m^2c^2}{\hbar^2}\phi - \frac{\lambda}{\hbar^2}(\phi^*\phi)\phi.$$

If we take the Minkowski metric, we get a Klein–Gordon type equation, that in the presence of an external potential U_{ext} (as in [26]) becomes

$$\frac{1}{c^2}\partial^2_{tt}\phi = \boldsymbol{\nabla}^2\phi - \frac{\lambda}{\hbar^2}(\phi^*\phi)\phi - \frac{m^2}{\hbar^2}\left(c^2 + \frac{2}{m}U_{\mathrm{ext}}\right)\phi. \tag{60}$$

Similarly, consider the metric of a weak gravitational field as given in [36], i.e.

$$\mathrm{d}s^2 = -(1+2\Phi)\mathrm{d}t^2 + (1-2\Phi)g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j,$$

where Φ is the gravitational potential; by assuming a static field we obtain

$$(1-2\Phi)\frac{1}{c^2}\partial^2_{tt}\phi = (1+2\Phi)\boldsymbol{\nabla}^2\phi - \frac{\lambda}{\hbar^2}(\phi^*\phi)\phi - \frac{m^2}{\hbar^2}\left(c^2 + \frac{2}{m}U_{\mathrm{ext}}\right)\phi. \tag{61}$$

Now, by substituting $\phi = \Psi\exp(-\mathrm{i}mc^2t/\hbar)$ into (60) we obtain the relativistic form of the GPE, given by

$$\mathrm{i}\hbar\partial_t\Psi - \frac{\hbar^2}{2mc^2}\partial^2_{tt}\Psi = \left(-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + \frac{\lambda}{2m}|\Psi|^2 + U_{\mathrm{ext}}\right)\Psi, \tag{62}$$

and similarly for (61). In the limit $c \to \infty$, from (60) (and from (61) taking $\Phi = 0$) we recover the non-relativistic GPE, given by

$$\mathrm{i}\hbar\,\partial_t\Psi = \left(-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + \frac{\lambda}{2m}|\Psi|^2 + U_{\text{ext}}\right)\Psi.$$

Since the two metrics above represent two instances of the general Riemannian treatment introduced in previous sections, the general setting of our extended equations proves amenable to the various specializations of the analogue gravity models considered in literature.

6.1. Hydrodynamics

As usual, the hydrodynamics description of the relativistic form (60) is given by applying the Madelung transformation $\Psi = \sqrt{\rho}\,\exp[\mathrm{i}(m/\hbar)\theta]$. By equating imaginary and real parts and identifying the individual contributions, we have the continuity equation

$$\partial_t\rho = \frac{1}{c^2}\partial_t(\rho\,\partial_t\theta) - \boldsymbol{\nabla}\cdot(\rho\boldsymbol{\nabla}\theta), \tag{63}$$

and the momentum equation

$$\partial_t\theta = -\frac{\hbar^2}{2m^2}\frac{1}{\sqrt{\rho}}\left(\frac{1}{c^2}\partial_{tt}^2\sqrt{\rho} - \boldsymbol{\nabla}^2\sqrt{\rho}\right) + \frac{1}{2}\left(\frac{1}{c^2}(\partial_t\theta)^2 - |\boldsymbol{\nabla}\theta|^2\right) - \frac{1}{m}\left(\frac{\lambda}{2m}\rho + U_{\text{ext}}\right). \tag{64}$$

Taking the gradient and setting $v = c^{-1}\partial_t\theta$ and $\boldsymbol{u} = \boldsymbol{\nabla}\theta$, the former becomes

$$\left(1 - \frac{v}{c}\right)\partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = -\frac{\hbar^2}{2m^2}\boldsymbol{\nabla}\left(\frac{\Box\sqrt{\rho}}{\sqrt{\rho}}\right) - \frac{\lambda}{2m^2}\boldsymbol{\nabla}\rho - \frac{1}{m}\boldsymbol{\nabla}U_{\text{ext}},$$

where $\Box = c^{-2}\partial_{tt} - \boldsymbol{\nabla}^2$ is the D'Alambertian operator. Similar results are obtained by applying the same procedure to (61), thus providing a set of dynamical laws for cosmological models based on condensate physics.

7. Concluding remarks

In this paper we have shown that the GPE hydrodynamic description can be extended to a generic Riemannian manifold. By doing so we have demonstrated (theorem 1) that the GPE can be written in the Euler form, and when we allow free expansion of the gas in absence of a trapping potential (an assumption that can be easily relaxed without consequences) it can take also the form of a Navier–Stokes type equation (theorem 2). In this case a new term, expressed by a density curvature vector, appears in the momentum equation: this term depends on the manifold local curvature and the gradients of the density distribution profile. By further analysis we have shown that when GPE is cast in the form of a conservation law for the momentum (theorem 3) it allows us to study some special, steady cases of interest. In the very special situation, provided by equation (54) (i) and (ii), a new form of an Einstein field equation is derived (equation (55)); in this circumstance the manifold Ricci scalar curvature attains a negative value, with Gaussian curvature given by $K = -16\pi a_s\rho$. A typical example of such geometry is represented by the Dini surface (of negative, constant curvature) of figure 1,

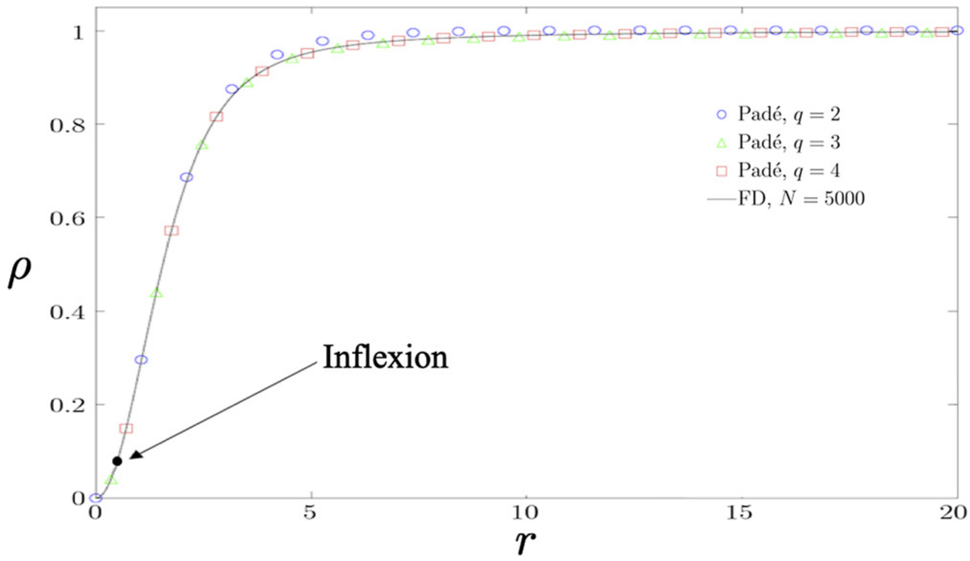

Figure 2. Density profile $\rho = \rho(r)$ plotted versus radial distance r from the nodal line; numerical solution (solid line) obtained by second-order finite differences on 5000 equispaced points compared with various Padé approximations given by q . Note the change of curvature at the inflexion point. Adapted from [37], Copyright (2018), with permission from Elsevier.

an interesting situation that encapsulates the cases in sections 5.1 and 5.2 above. The relevance of these results for cosmological models is highlighted by the relativistic forms of GPE [26, 36] derived above, thus providing rigorous grounds for further investigations and laboratory experiments to test analogue models of black hole radiation based on supersonic phonon flows in condensates.

The advantage of the analytical results presented here is to provide a dynamical description of the GPE evolution in more general contexts and to highlight the interplay of local geometry and gas density, allowing numerical investigations of the stationary configurations discussed in sections 5.1 and 5.2 above. Another key feature of this study is to emphasize the role of the density distribution profile in the density curvature vector. This term becomes particularly relevant when we approach the defect region, because density must go rapidly to zero towards the singularity. It is well-known that in this region the density profile gradient must go through an inflexion point (see figure 2) [38], with outwardly pointing gradient everywhere positive. In the case of negative scalar curvature (as in equation (56)), the density curvature vector $\boldsymbol{\mathcal{E}}$ becomes increasingly negative in the outer part of the healing region, thus contributing to the natural trapping of acoustic waves. Since acoustic waves in condensates are analogues to light waves in black hole theory, this mechanism seems to enforce the analogy between defect properties in condensates and black hole dynamics in cosmology.

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Data availability statement

All data that support the findings of this study are included within the article (and any suppl ementary files).

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