IOP Publishing | The Japan Society of Fluid Mechanics Fluid *Dynamics* Research

Fluid Dyn. Res. **46** (2014) 061412 (9pp) doi:10.1088/0169-5983/46/6/061412

The Jones polynomial as a new invariant of topological fluid dynamics

Renzo L Ricca[1] **and Xin Liu**[1,2]

[1] Department of Mathematics and Applications, University of Milano-Bicocca, Via Cozzi 55, 20125 Milano, Italy
[2] Beijing-Dublin International College and Institute of Theoretical Physics, Beijing University of Technology, 100 Pingleyuan, Beijing 100124, People's Republic of China

E-mail: renzo.ricca@unimib.it

Received 25 October 2013, revised 29 August 2014
Accepted for publication 31 August 2014
Published 16 October 2014

Communicated by Y Fukumoto

Abstract

A new method based on the use of the Jones polynomial, a well-known topological invariant of knot theory, is introduced to tackle and quantify topological aspects of structural complexity of vortex tangles in ideal fluids. By re-writing the Jones polynomial in terms of helicity, the resulting polynomial becomes then function of knot topology and vortex circulation, providing thus a new invariant of topological fluid dynamics. Explicit computations of the Jones polynomial for some standard configurations, including the Whitehead link and the Borromean rings (whose linking numbers are zero), are presented for illustration. In the case of a homogeneous, isotropic tangle of vortex filaments with same circulation, the new Jones polynomial reduces to some simple algebraic expression, that can be easily computed by numerical methods. This shows that this technique may offer a new setting and a powerful tool to detect and compute topological complexity and to investigate relations with energy, by tackling fundamental aspects of turbulence research.

1. Vorticity localization in classical and quantum fluids

In recent years there has been overwhelming evidence that vorticity tends to get organized into coherent vortex filaments and tubes (the 'sinews of turbulence'), in both classical and quantum fluids. Numerical and observational experiments (Golov and Walmsley 2009, Uddin *et al* 2009) on the insurgence and permanence of isotropic turbulence demonstrate that this is characterized by relatively long-lived complex tangles of filaments, that are randomly

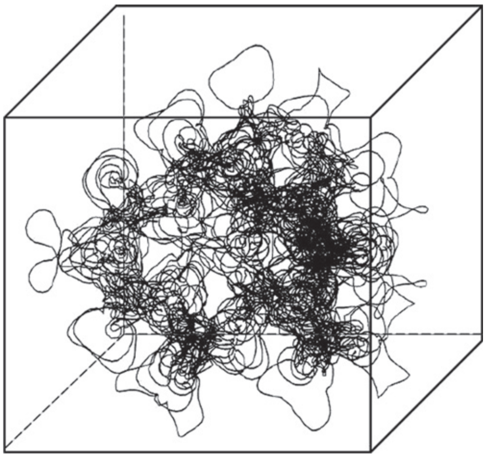

Figure 1. Tangle of superfluid vortex filaments produced by the action of a superimposed ABC flow. Visualization produced by numerical simulation of the governing equations (adapted with permission from Barenghi *et al* 2001, Copyright 2001 Elsevier).

distributed in the bulk of the fluid, before final dissipation. Such evidence has been further confirmed (Baggaley *et al* 2012) by recent enhanced visualization techniques and refined numerical codes, coupled with increased computational power and modern diagnostic methods (Hauser *et al* 2007). The most recent analyses show that indeed there is a striking similarity between structure and energy spectra of classical and quantum turbulence, even though the fundamental mechanisms are quite different. In both cases vortex tangles are formed by an intricate spatial network of filaments (see figure 1), that in the case of superfluids may be truly considered material vortex lines (Barenghi *et al* 2001). Vortex knots and links have been produced by direct numerical simulations under various conditions (see, for instance, Ricca and Berger (1996) and references therein) and have been studied numerically as evolution solutions to integrable equations (Ricca *et al* 1999). Their existence as mathematical solutions to the Euler equations has been established recently by Enciso and Peralta (2012) and, remarkably, vortex knots and links in water have been produced by laboratory experiments by Kleckner and Irvine (2013). Structural complexity methods based on the use of geometric and topological information (Ricca 2009, Moffatt *et al* 2013), are therefore particularly appropriate to interpret and quantify complex features in terms of physical properties.

With this paper we want to introduce the Jones polynomial, a well-known topological invariant of knot theory, as a new, powerful invariant of topological fluid mechanics. In particular, we want to show that the Jones polynomial of fluid knots may represent a useful tool to investigate physical properties of complex flows. Since this invariant is defined by means of the kinetic helicity of a vortex tangle, we need some preliminary information about the computation of helicity. This is done in the following section. We shall then introduce the Jones polynomial (section 3) and, by some elementary examples, compute the polynomial of some simple configurations for illustration (section 4). In section 5 we shall introduce the Jones polynomial in terms of helicity as a new invariant of topological fluid mechanics and finally, in section 6, show how this can be used to investigate aspects of structural complexity for isotropic turbulent flows.

2. Kinetic helicity of vortex tangles

In general kinetic helicity (Saffman 1992) is defined by

$$H \equiv \int_{\Omega} \mathbf{u} \cdot \omega \, \mathrm{d}^3\mathbf{x} \ , \tag{1}$$

where $\mathbf{u}$ is the velocity field, defined on an unbounded domain of $\mathbb{R}^3$, where $\omega = \nabla \times \mathbf{u}$ is the vorticity defined on a sub-domain Ω of $\mathbb{R}^3$. For simplicity we assume $\nabla \cdot \mathbf{u} = 0$ everywhere, and we request $\omega \cdot \hat{\mathbf{n}} = 0$ on $\partial\Omega$, where $\hat{\mathbf{n}}$ is orthogonal to $\partial\Omega$, with $\nabla \cdot \omega = 0$. In this paper we limit ourselves to the case of ideal fluid flows, and consider a tangle $\mathcal{T}$ of vortex filaments given by a collection $\mathcal{T} = \cup_i \mathcal{K}_i$ of N thin, vortex knots $\mathcal{K}_i$ ($i = 1,\ldots,N$). These filaments are centered on simple, smooth, *oriented* curves, with orientation induced naturally by the vorticity vector defined on each filament.

The topological interpretation of helicity in terms of Gauss linking number was first established by Moffatt (1969), and subsequently extended by Moffatt and Ricca (1992) (see also Ricca and Moffatt 1992) to include the contribution from self-helicity; hence, for a tangle of N vortex filaments of circulation κ_i, the kinetic helicity $H = H(\mathcal{T})$ can be written in the alternative form

$$H(\mathcal{T}) = \sum_i \kappa_i^2 SL_i + 2\sum_{ij} \kappa_i \kappa_j Lk_{ij} \ , \tag{2}$$

where SL_i is the (Călugăreanu-White) self-linking number of the ith vortex, and Lk_{ij} the (Gauss) linking number of $\mathcal{K}_i$ and $\mathcal{K}_j$.

In a number of cases the topological computation of helicity given by the equation above gives wrong results, because of limitations of the Gauss linking number formula to compute correct values of topological complexity. The cases of the Whitehead link and the Borromean rings, discussed in some detail in section 4 below, are typical examples of topologically distinct links, whose linking number is identically zero. Quite simply this means that the linking number is not sufficient to detect unambiguously different link types. In order to address this problem new tools, such as knot polynomials, have been introduced to provide a finer topological characterization of knots and links.

3. Tackling topological complexity by knot polynomials: the Jones polynomial

Knot polynomials belong to a special class of polynomials (of Laurent type), written in terms of a dummy variable τ, that has no physical meaning. Each of them provides a topological invariant of the knot/link considered through a (possibly) unique characterization of its topology. The first type of knot polynomial was introduced by J.W. Alexander in 1923, but it was only in 1969 that J. Conway showed that the Alexander polynomial of a knot can be derived by using a recursive relation (called *skein relation*) based on elementary computations performed on each crossing site of the knot diagram (Kauffman 1987, 2001). Other polynomials followed suit, including the Jones polynomial, introduced by V.F.R. Jones in 1984, that is now a familiar tool among experts. Data bases (such as `KnotAtlas` and `KnoInfo`) on knots and links and dedicated software (such as `KnotPlot` and `LinKnot`) are nowadays freely available online.

Given a knot/link in space, we can always determine its diagram by considering the indented projection onto a plane. This is done by allowing small positive/negative indentations (shown by over- and under-passes) of the incident strands to keep track of the original topology. This diagram is not unique and it depends on the projection direction: see, for

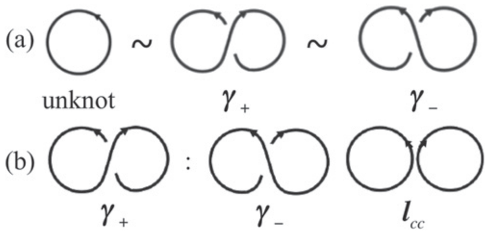

Figure 2. (a) The standard circle (unknot) is topologically equivalent to the figure-of-eight with an (apparent) over-crossing (γ_+), or an under-crossing (γ_-). (b) The Jones polynomial V of two unlinked loops $\boldsymbol{l}_{cc}$ is obtained by applying (4) to γ_+.

instance, the example shown in figure 2 (a) , where a standard circle (the unknot) can be seen to form a figure-of-eight with an (apparent) over-crossing (γ_+), or under-crossing (γ_-).

The significance of the skein relation was not realized until the discovery of the Jones polynomial by V.F.R. Jones (1984). Let us introduce the skein relation for this polynomial; this is given by a set of rules that prescribe how a 'state' of a particular crossing site (represented, for example, by an over-crossing) of the knot diagram can be related to its opposite (i.e. an under-crossing) and their 'smoothing', given by two parallel strands (a non-crossing). The recursive application of the skein relation to each crossing allows the computation of the polynomial for the given knot. Hence, let $\mathcal{K}$ be a knot or a link, and $V(\mathcal{K})$ the Jones polynomial of $\mathcal{K}$. Formally, the skein relation for $V(\mathcal{K})$ is given by the following set of rules:

$$\text{(i)} \quad V(\circ) = 1, \tag{3}$$

$$\text{(ii)} \quad \tau^{-1} V(\nwarrow\!\!\!\!\nearrow) - \tau V(\nearrow\!\!\!\!\nwarrow) = \left(\tau^{\frac{1}{2}} - \tau^{-\frac{1}{2}}\right) V(\,)(\,). \tag{4}$$

The first rule states that the Jones polynomial of the unknot (or any topologically equivalent presentation of the unknot) $\circ$ is equal to 1; hence

$$V(\circ) = V\left(\gamma_+\right) = V\left(\gamma_-\right) = 1. \tag{5}$$

Since the three presentations of figure 2(a) are all topologically equivalent to the unknot, they all have the same Jones polynomial, and this is equal to 1. The second rule establishes a relation between an over-crossing, an under-crossing and a non-crossing via a dummy variable τ. This second rule must be applied recursively to each single crossing site at a time. For illustration, let us apply this second rule to γ_+ (see figure 2(b)); since $V(\gamma_+) = V(\gamma_-) = 1$, we obtain the Jones polynomial $V(l_{cc})$ of two unlinked loops, that is

$$V\left(\boldsymbol{l}_{cc}\right) = -\tau^{-\frac{1}{2}} - \tau^{\frac{1}{2}}. \tag{6}$$

By a straightforward computation we can show that, in general, for N loops we have

$$V\left(\boldsymbol{l}_{c,\ldots,c}\right) = \left(-\tau^{-\frac{1}{2}} - \tau^{\frac{1}{2}}\right)^{N-1}. \tag{7}$$

Note that indeed $V(\boldsymbol{l}_{c,\ldots,c}) \neq V(\circ)$ and orientation does not matter. A physical interpretation of the skein relation is given in terms of a correspondence between field change and field line curvature and curl (see Kauffman 2001). Some simple configurations and their computed Jones polynomial are shown in figure 3.

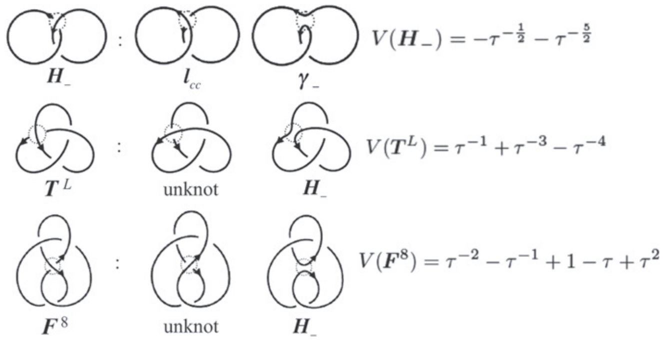

Figure 3. The Jones polynomial of the (negative) Hopf link $V(\boldsymbol{H}_-)$, left-handed trefoil knot $V(\boldsymbol{T}^L)$, and figure-of-eight knot $V(\boldsymbol{F}^8)$.

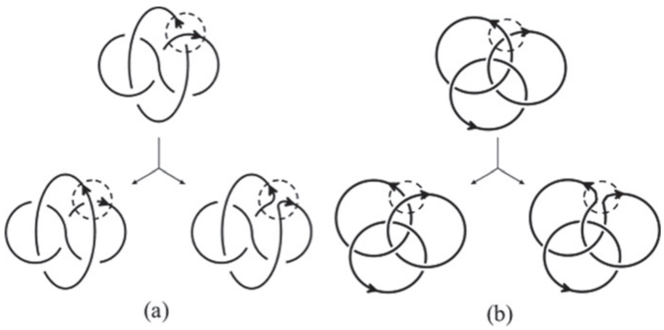

Figure 4. Reduction schemes: (a) the Whitehead link $\boldsymbol{W}_+$ (top diagram) gives the negative Hopf link $\boldsymbol{H}_-$ and the left-handed trefoil knot $\boldsymbol{T}^L$; (b) the Borromean rings $\boldsymbol{B}$ give the positive Hopf link $\boldsymbol{H}_+$ and the Whitehead link $\boldsymbol{W}$.

4. Computation of the Jones polynomial for the Whitehead link and Borromean links

It is well known that there are many link types that are inseparably linked, even though their linking number is zero (as for two unlinked loops): the Whitehead link and the Borromean rings are two typical examples. For such systems the linking number computation fails to give a correct value, and therefore the helicity computation in terms of linking numbers is bound to give wrong results. Let us compute the Jones polynomial instead.

4.1. Whitehead link

First let us consider the Whitehead link $\boldsymbol{W}_+$ shown in figure 4a, top diagram. By applying the skein relation (4) to the encircled crossing site, we can see that $\boldsymbol{W}_+$ reduces to the negative Hopf link $\boldsymbol{H}_-$ and the left-hand trefoil knot $\boldsymbol{T}^L$, whose diagrams are shown at the bottom of figure 4(a). Each diagram represents the knot type obtained by switching the selected crossing into its opposite and by smoothing it into the non-crossing. Algebraically, we have

$$\tau^{-1}V\left(W_{+}\right) - \tau V\left(\boldsymbol{H}_{-}\right) = \left(\tau^{\frac{1}{2}} - \tau^{-\frac{1}{2}}\right)V\left(\boldsymbol{T}^{L}\right). \tag{8}$$

Now, by using the Jones polynomial computed for $\boldsymbol{H}_{-}$ and $\boldsymbol{T}^{L}$ (see figure 3), we can derive $V(\boldsymbol{W}_{+})$. The same analysis can be applied to $\boldsymbol{W}_{-}$, which gives the positive Hopf link $\boldsymbol{H}_{+}$ and the figure-of-eight knot $\boldsymbol{F}^{8}$, whose diagram and polynomial are shown in figure 3. Similarly as above, by substituting the relative algebraic expressions we can prove that $V(\boldsymbol{W}_{+}) = V(\boldsymbol{W}_{-}) = V(\boldsymbol{W})$, showing that the two oriented knots are actually the same knot type, whose polynomial is given by

$$V(\boldsymbol{W}) = \tau^{-\frac{7}{2}} - 2\tau^{-\frac{5}{2}} + \tau^{-\frac{3}{2}} - 2\tau^{-\frac{1}{2}} + \tau^{\frac{1}{2}} - \tau^{\frac{3}{2}}. \tag{9}$$

4.2. Borromean rings

Similarly for the Borromean rings $\boldsymbol{B}$. The diagram of $\boldsymbol{B}$ is shown on top of figure 4(b). By applying the skein relation (4) to the encircled crossing site, $\boldsymbol{B}$ reduces to the positive Hopf link $\boldsymbol{H}_{+}$ and the Whitehead link $\boldsymbol{W}$. Thus, we have

$$V(\boldsymbol{B}) = -\tau^{-3} + 3\tau^{-2} - 2\tau^{-1} + 4 - 2\tau + 3\tau^{2} - \tau^{3}\ . \tag{10}$$

5. The Jones polynomial as a new invariant of topological fluid mechanics

By considering vortex filaments as material lines, we can reduce the helicity integral (1) to a discrete sum of line integrals (Barenghi *et al* 2001), given by

$$H(\mathcal{T}) = \sum_{i=1}^{N} \kappa_i \oint_{\mathcal{K}_i} \mathbf{u} \cdot \mathrm{d}\mathbf{l}, \tag{11}$$

where now $\mathbf{u}$ denotes the velocity induced by the Biot-Savart law. On the other hand, by applying the Călugăreanu–White formula (Moffatt and Ricca 1992, Ricca and Moffatt 1992) to a single tangle component (and by dropping the index), we can show that (2) reduces to the sum of writhe and twist helicity, given by

$$H(\mathcal{K}) = \kappa^2 SL = \kappa^2 (Wr + Tw), \tag{12}$$

where the writhing number Wr and the total twist number Tw are geometric properties of the physical knot $\mathcal{K}$.

By making use of a plausible statistical hypothesis on the state decomposition of crossing sites (equivalent to an ergodicity assumption on the state ensemble) and by using (12), Liu and Ricca (2012) proved the following result:

Theorem. *(Liu and Ricca 2012): Let $\mathcal{K}$ denote a vortex knot. If the helicity of $\mathcal{K}$ is $H = H(\mathcal{K})$, then*

$$e^{H(\mathcal{K})} = e^{\oint_{\mathcal{K}} \mathbf{u}\cdot \mathrm{d}\mathbf{l}}, \tag{13}$$

appropriately re-scaled, satisfies (with a plausible statistical hypothesis) the skein relations of the Jones polynomial $V = V(\mathcal{K})$.

Remark 1. Helicity, defined by (1) or by its limiting form (11), is an invariant of topological fluid mechanics under ideal diffeomorphisms of fluid flows.

Remark 2. To make sense of $e^{H(\mathcal{K})}$, $H(\mathcal{K})$ must be written in dimensionless form, by a re-normalization with respect to some reference value of the circulation.

Remark 3. Proof of the theorem relies on the ergodic assumption that all possible state decompositions of crossing sites have equal probability to occur.

Remark 4. By setting

$$\tau = e^{-4\lambda H\left(\gamma_+\right)}, \qquad \lambda \in [0,\, 1], \tag{14}$$

the skein relations (3) and (4) generate the Jones polynomial of the vortex knot/link. Here $H(\boldsymbol{\gamma}_+)$ is the helicity of a reference unknotted vortex loop and λ is a parameter that takes into account the uncertainty associated with the writhe value of $\boldsymbol{\gamma}_+$. For a large collection of knots $H(\boldsymbol{\gamma}_+)$ is interpreted as a background mean twist field given by the average value of the helicity of the unknots present in the tangle.

The Jones polynomial of a vortex knot becomes thus function of the topology of the knot $\mathcal{K}$ as well as of the vortex circulation κ, and in general for an N-component link $V = V(\mathcal{K}_i;\, \kappa_i)$. This naturally extends the dual dependence of helicity on linking numbers and circulation, given by $H = H(SL_i,\, Lk_{ij};\, \kappa_i)$.

6. Applications to homogeneous vortex tangles

For practical purposes let us consider a homogeneous, isotropic tangle of superfluid filaments, where all vortices have same circulation κ (see, for example, the tangle of figure 1). It is reasonable to assume

$$\bar{\lambda} = \langle\lambda\rangle = \frac{1}{2}, \quad \left\langle H\left(\gamma_+\right)\right\rangle = \frac{\kappa^2}{2}, \tag{15}$$

where angular brackets denote average values over the whole ensemble of vortices. For superfluid vortex filaments, that have no internal structure, the average helicity could be related to the average value of the normalized total torsion of the whole tangle (twist helicity), or the average value of the writhing number (writhe helicity) (Moffatt and Ricca 1992). By normalizing helicity with respect to this κ (or, equivalently, by setting $\kappa = 1$) from (14) we have

$$\tau = e^{-1}. \tag{16}$$

Thus, in the case of two unlinked vortex rings of equal circulation the re-normalized Jones polynomial becomes

$$V\left(\boldsymbol{l}_{cc}\right) = -e^{\frac{\kappa^2}{2}}\left(1 + e^{-\kappa^2}\right) \to V\left(\boldsymbol{l}_{cc}\right) = -e^{\frac{1}{2}}\left(1 + e^{-1}\right). \tag{17}$$

Similarly for some of the standard configurations we considered earlier, for which we have:

$$V(\circ) = V\left(\gamma_+\right) = V\left(\gamma_-\right) = 1, \tag{18}$$

$$V\left(\boldsymbol{l}_{c,\ldots,c}\right) = \left[-\, e^{\frac{1}{2}}\left(1 + e^{-1}\right)\right]^{N-1}, \qquad (N \text{ rings}), \tag{19}$$

$$V\left(\boldsymbol{H}_{+}\right) = -e^{-\frac{1}{2}}\left(1 + e^{-2}\right), \tag{20}$$

$$V(\boldsymbol{H}_{-}) = -e^{\frac{1}{2}}\left(1 + e^{2}\right) \tag{21}$$

$$V\left(\boldsymbol{T}^{L}\right) = e + e^{3} - e^{4}, \tag{22}$$

$$V\left(\boldsymbol{T}^{R}\right) = e^{-1} + e^{-3} - e^{-4}. \tag{23}$$

$$V\left(\boldsymbol{F}^{8}\right) = e^{2} - e + 1 - e^{-1} + e^{-2}, \tag{24}$$

$$V(\boldsymbol{W}) = e^{\frac{7}{2}} - 2e^{\frac{5}{2}} + e^{\frac{3}{2}} - 2e^{\frac{1}{2}} + e^{-\frac{1}{2}} - e^{-\frac{3}{2}}, \tag{25}$$

$$V(\boldsymbol{B}) = -e^{3} + 3e^{2} - 2e + 4 - 2e^{-1} + 3e^{-2} - e^{-3}. \tag{26}$$

More complex computations could be implemented in numerical codes for a real-time diagnostics of intricate networks of vortical flows. With reference to the tangle of figure 1, for instance, `KnotPlot` can be used to perform a topological analysis of the computed data, by detecting knots and links present in the system at a given time. A time-dependent spectral distribution of structures, in terms of increasing topological complexity given by the number of open filaments, unlinked unknotted loops, trefoil knots, Hopf links, etc., can be produced. The associated polynomials can then be computed and related to kinetic energy. Since reconnection events and dissipation mechanisms change vortex topology, by repeating this process adaptively at subsequent times, different number of unknots and different types of knots and links will emerge. Consequently, new relations between topological complexity and dynamical properties of the system will be found. This process can be encapsulated in an automatic adaptive analysis to be implemented by appropriate diagnostic toolkits in direct numerical simulations of complex vortex flows. Work is in progress to apply and extend this approach to real fluid contexts, by including other knot polynomial invariants and, in a different direction, topological changes due to vortex reconnection and viscous dissipation. Our hope is that in the long term this approach could provide a novel setting for new, useful tools to tackle long-standing problems in turbulence research.

Acknowledgements

We wish to thank the Isaac Newton Institute for Mathematical Sciences for the kind hospitality and support provided during our stay in Cambridge (UK) in 2012, where part of this work was done.

References

Baggaley A W, Barenghi C F, Shukurov A and Sergeev Y A 2012 Coherent vortex structures in quantum turbulence *Europhys. Lett.* **98** 26002

Barenghi C F, Ricca R L and Samuels D C 2001 How tangled is a tangle? *Physica* D **157** 197–206

Enciso A and Peralta-Salas D 2012 Knots and links in steady solutions of the Euler equation *Ann. Math.* **175** 345–67

Golov A I and Walmsley P M 2009 Homogeneous turbulence in superfluid 4He in the low-temperature limit: experimental progress *J. Low Temp. Phys.* **156** 51–70

Hauser H, Hagen H and Theisel H 2007 *Topology-based Methods in Visualization* (Heidelberg: Springer-Verlag)

Jones V F R 1987 Hecke algebra representations of braid groups and link polynomials *Ann. Math.* **126** 335–88

Kauffman L H 1987 *On Knots* (Princeton: Princeton University Press)

Kauffman L H 2001 *Knots and Physics* (Singapore: World Scientific Publishers)

Kleckner D and Irvine W T M 2013 Creation and dynamics of knotted vortices *Nature Physics* **9** 253–8 See also: http://nature.com/news/physicists-twist-water-into-knots-1.12534

KnotAtlas: http://katlas.math.toronto.edu/wiki/Main_Page

KnotInfo: http://indiana.edu/~knotinfo

KnotPlot: http://www.knotplot.com

LinKnot: http://www.mi.sanu.ac.rs/vismath/linknot/index.html

Liu X and Ricca R L 2012 The Jones polynomial for fluid knots from helicity *J. Phys. A: Math. & Theor.* **45** 205501

Liu X and Ricca R L 2013 Tackling fluid structures complexity by the Jones polynomial *Topological Fluid Dynamics: Theory and Applications Procedia IUTAM* vol 7 (Dordrecht: Elsevier) p 175

Moffatt H K 1969 The degree of knottedness of tangled vortex lines *J. Fluid Mech.* **35** 117–29

Moffatt H K, Bajer K and Kimura Y 2013 *Topological Fluid Dynamics: Theory and Applications Procedia IUTAM* vol 7 (Dordrecht: Elsevier)

Moffatt H K and Ricca R L 1992 Helicity and the Călugăreanu invariant *Proc. R. Soc.* A **439** 411–29

Ricca R L and Berger M A 1996 Topological ideas in fluid mechanics *Phys. Today* **49** 24–30

Ricca R L, Samuels D C and Barenghi C F 1999 Evolution of vortex knots *J. Fluid Mech.* **391** 29–44

Ricca R L 2001 *An Introduction to the Geometry and Topology of Fluid Flows NATO ASI Series II* vol 47 (Dordrecht: Kluwer Academic Publishers)

Ricca R L 2009 Structural complexity and dynamical systems *Lectures on Topological Fluid Mechanics Springer-CIME Lecture Notes in Mathematics* vol 1973 (Heidelberg: Springer-Verlag) p 169

Ricca R L and Moffatt H K 1992 The helicity of a knotted vortex filament *Topological Aspects of the Dynamics of Fluids and Plasmas* (Dordrecht: Kluwer Academic) p 225

Saffman P G 1992 *Vortex Dynamics* (Cambridge: Cambridge University Press)

Uddin M A, Oshima N, Tanahashi M and Miyauchi T 2009 A study of the coherent structures in homogeneous isotropic turbulence *Proc. Pakistan Acad. Sci.* **46** 145–58