

Topological Fluid Dynamics: Theory and Applications

# Tackling fluid structures complexity by the Jones polynomial

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## Abstract

By making simple, heuristic assumptions, a new method based on the derivation of the Jones polynomial invariant of knot theory to tackle and quantify structural complexity of vortex filaments in ideal fluids is presented. First, we show that the topology of a vortex tangle made by knots and links can be described by means of the Jones polynomial expressed in terms of kinetic helicity. Then, for the sake of illustration, explicit calculations of the Jones polynomial for the left-handed and right-handed trefoil knot and for the Whitehead link via the figure-of-eight knot are considered. The resulting polynomials are thus function of the topology of the knot type and vortex circulation and we provide several examples of those. While this heuristic approach extends the use of helicity in terms of linking numbers to the much richer context of knot polynomials, it gives also rise to new interesting problems in mathematical physics and it offers new tools to perform real-time numerical diagnostics of complex flows.

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*Keywords:* vortex knots and links; helicity; Jones polynomial; topological fluid dynamics; structural complexity.

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## 1. Introduction

The purpose of this paper is to introduce and elucidate a novel technique to quantify topological information of complex tangles of vortex filaments in ideal fluids by means of the Jones polynomial, a powerful invariant of knot theory. This new approach, based on some simple, heuristic assumptions, involves the implementation of kinetic helicity of vortex flows by an appropriate transformation. Helicity plays a fundamental rôle in fluid mechanics, being an invariant of the Euler equations and a robust quantity of the dissipative Navier-Stokes equations. Kinetic helicity is defined by

$$H \equiv \int_{\Omega} \mathbf{u} \cdot \boldsymbol{\omega} d^3\mathbf{x}, \quad (1)$$

where  $\mathbf{u}$  is the velocity field, defined on an unbounded domain of  $\mathbb{R}^3$ , and  $\boldsymbol{\omega} = \nabla \times \mathbf{u}$  is the vorticity, defined on a sub-domain  $\Omega$ . For simplicity we assume  $\nabla \cdot \mathbf{u} = 0$  everywhere, and we request  $\boldsymbol{\omega} \cdot \hat{\mathbf{n}} = 0$  on  $\partial\Omega$ , where  $\hat{\mathbf{n}}$  is

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orthogonal to  $\partial\Omega$ , with  $\nabla \cdot \boldsymbol{\omega} = 0$ . In this paper we consider a tangle of vortex filaments given by a collection  $\cup_i \mathcal{K}_i$  of  $N$  thin vortex knots  $\mathcal{K}_i$  ( $i = 1, \dots, N$ ) in  $\mathbb{R}^3$ . These filaments are centered on simple, smooth, *oriented* curves.

The topological interpretation of helicity in terms of linking numbers was originally established by Moffatt [1], and subsequently extended by Moffatt & Ricca [2], by the equation

$$H(\mathcal{K}) = \sum_i \kappa_i^2 Lk_i + 2 \sum_{ij} \kappa_i \kappa_j Lk_{ij} , \tag{2}$$

where  $\kappa_i$  denotes the  $i$ -th vortex circulation,  $Lk_i$  the (Călugăreanu-White) self-linking number of the  $i$ -th vortex, and  $Lk_{ij}$  the (Gauss) linking number of  $\mathcal{K}_i$  and  $\mathcal{K}_j$ . While this formula proves functional in many physical situations, it has limitations, when the linking numbers fail to capture essential topology, as in the case of the Whitehead link and the Borromean rings. A finer topological description is therefore needed.

### 2. The Jones polynomial in terms of helicity of oriented knots

In analogy with work done in topological quantum field theory [4, 5], finer characterization of knot topology may be provided by knot polynomial invariants derived from classical field theory. In the context of topological fluid mechanics this can be done by noticing that helicity can be interpreted as an abelian Chern-Simons (CS) action, and that, by using some heuristic assumptions, the transformation  $e^{H(\mathcal{K})}$  (or, equivalently, by an appropriately re-scaled new variable  $\tau$ ) can be shown to realize the skein relations of a powerful knot polynomial invariant, the Jones polynomial  $V$  of the knot  $\mathcal{K}$  (hence, of the tangle  $\cup_i \mathcal{K}_i$ ).

For simplicity, let us consider an isolated vortex knot  $\mathcal{K}$  in an ideal fluid. Vortex knots in superfluid helium at very low temperature may provide a good practical example. In this case, due to the thinness of the vortex and the fact that vorticity (directed along the unit tangent to the vortex centerline) is essentially a constant (actually a  $\delta$ -Dirac function) over the vortex cross-section, it is possible to reduce (1) to a loop integral [3], that is

$$H = \kappa \oint_{\mathcal{K}} \mathbf{u} \cdot d\mathbf{l} , \tag{3}$$

where now  $\mathbf{u}$  denotes the vortex velocity induced by the Biot-Savart law. On the other hand, for a single tangle component equation (2) reduces to the well-known contributions due to the Călugăreanu-White formula [2], i.e. (by dropping the index)

$$H(\mathcal{K}) = \kappa^2 Lk = \kappa^2 (Wr + Tw) , \tag{4}$$

where helicity gets decomposed in terms of writhe ( $Wr$ ) and twist ( $Tw$ ) contributions.

The Jones polynomial  $V = V(\mathcal{K})$  is a topological invariant of the knot  $\mathcal{K}$  [6], function of a dummy variable (say  $\tau$ ), that in general has no physical meaning: thus  $V(\mathcal{K}) = V_\tau(\mathcal{K})$ . In our case  $\mathcal{K}$  is a vortex knot, carrying topological as well as dynamical information during its motion. This dual property is well captured by combining the two equations (3) and (4). Hence, in topological fluid dynamics to prove that the Jones polynomial is indeed an isotopy invariant of a vortex knot (or a tangle), we must write  $\tau$  in terms of helicity (since also helicity is an isotopy invariant), and show that that  $\tau$  is consistent with the derivation of the skein relations, that define the Jones polynomial [7].

Indeed, by using (3) and (4) and the transformation  $e^H \rightarrow t^H \rightarrow \tau$ , we have the following:

**Theorem (Liu & Ricca, 2012):** *Let  $\mathcal{K}$  denote a vortex knot (or an  $N$ -component link) of helicity  $H = H(\mathcal{K})$ . Then*

$$t^{H(\mathcal{K})} = t^{\oint_{\mathcal{K}} \mathbf{u} \cdot d\mathbf{l}} , \tag{5}$$

*appropriately re-scaled, satisfies (with a plausible statistical hypothesis) the skein relations of the Jones polynomial  $V = V(\mathcal{K})$ .*

A detailed proof of the above theorem is given by Liu and Ricca (2012), and it is based on a two-step approach: first the derivation of the Kauffman bracket polynomial [7] for the *unoriented* knot, by deducing the corresponding skein

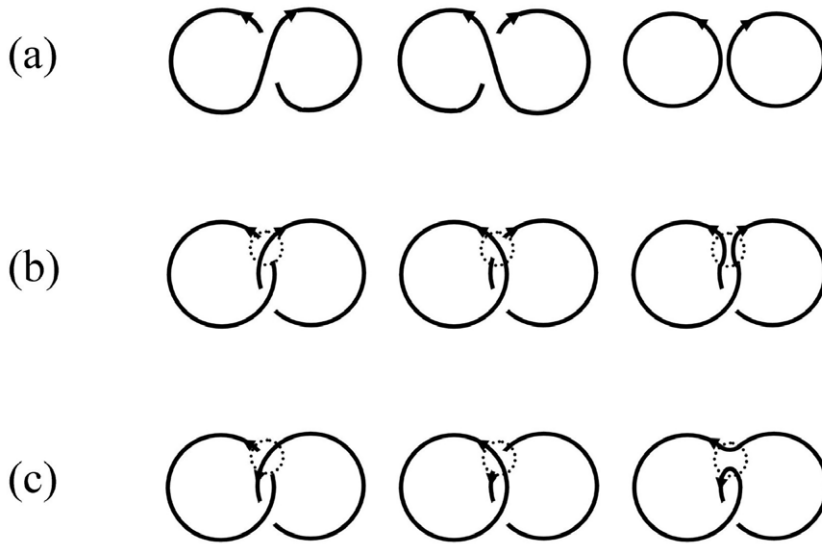


Fig. 1. (a) Writhe  $\gamma_+$ ,  $\gamma_-$  and disjoint union of two trivial circles  $l_{cc}$ . (b) Hopf link  $H_+$  with crossing  $+1$ , disjoint union of circles  $l_{cc}$  and writhe  $\gamma_+$ . (c) Hopf link  $H_-$  with crossing  $-1$ , disjoint union of circles  $l_{cc}$  and writhe  $\gamma_-$ . Note that the orientation of any number of disjoint circles does not influence the polynomial.

relations in terms of  $t^{H(\mathcal{K})}$ ; then, by including knot orientation, the derivation of the Jones polynomial by deducing the skein relations in terms of the new variable

$$\tau = t^{-4\lambda H(\gamma_+)}, \quad \lambda \in [0, 1], \tag{6}$$

where  $\lambda$  takes into account the uncertainty associated with the writhe value of  $\gamma_+$  (see the first diagram of Figure 1a) and  $H(\gamma_+)$  denotes the helicity associated with  $\gamma_+$  [8]. In the first part, the derivation of the Kauffman bracket polynomial is done by assuming heuristically that the splitting decompositions of the unoriented crossing state have equal possibility to occur (see Figure 2). This idea relies on the assumption of dealing with a large collection of different knot presentations, that is consistent with a full combinatorial approach to tangle complexity; this equal possibility implicitly attributed to each possible decomposition can be justified as an average value for a large number of events. If this were not the case, surely new interesting problems associated with the skewness of the statistics would arise. The skein relations are then standardly derived by a technique called *local path-addition*, that consists of computing crossing states according to the analysis of the elementary states given by the over-crossing  $L_+ = \nearrow \searrow$ , the under-crossing  $L_- = \searrow \nearrow$ , and the disjoint union with a trivial circle  $\uparrow \sqcup \bigcirc$ . The skein relation of the Jones polynomial are obtained thus:

$$V(\bigcirc) = 1, \tag{7}$$

$$\tau^{-1}V(\searrow \nearrow) - \tau V(\nearrow \searrow) = (\tau^{\frac{1}{2}} - \tau^{-\frac{1}{2}}) V(\searrow \nearrow). \tag{8}$$

We should remark here that local path-additions are purely *mathematical* operations, performed *virtually* on the knot strands, the only purpose being simply the mathematical derivation of the polynomial terms, that give rise to the desired polynomial invariant. No actual physical process is therefore involved.

### 3. Computation of the Jones polynomial for the left- and right-handed trefoil knots and for the Whitehead link

For the sake of illustration we compute the Jones polynomial by considering the following examples.

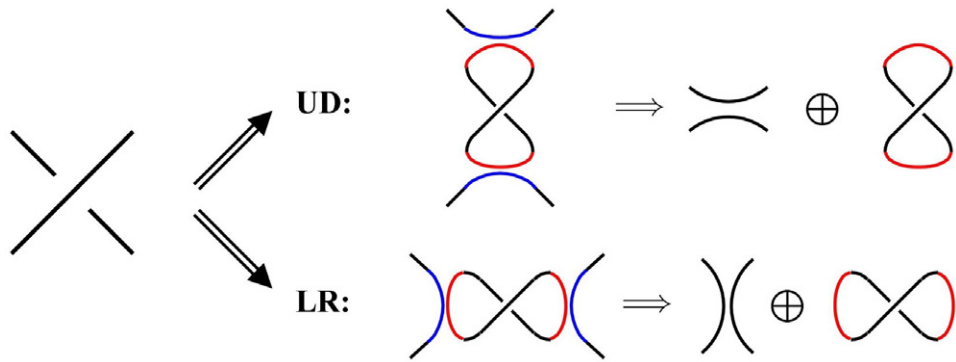


Fig. 2. The unoriented crossing on the left can be split by adding and subtracting local paths following the up-down (UD) or the left-right (LR) scheme above. Both decompositions are assumed to contribute equally to the calculation of  $t^H$ .

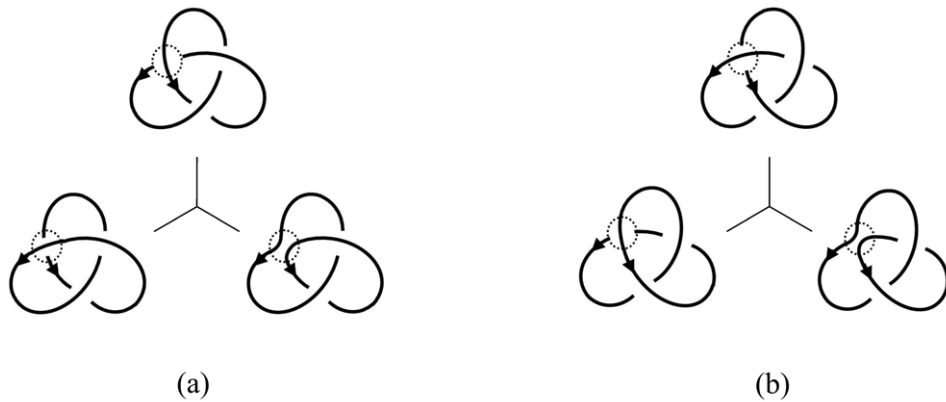


Fig. 3. (a) Left-handed and (b) right-handed trefoil knots (top diagrams) decomposed by applying standard reduction techniques (local path-addition) on crossing sites. Their Jones polynomials are obtained by analyzing the elementary states given by the diagrams of Figure 1.

### 3.1. Left-handed and right-handed trefoil knots

The left-handed trefoil knot  $T^L$  and the right-handed trefoil knot  $T^R$  are shown by the top diagram of Figure 3a and 3b, respectively. By re-arranging (8), we can convert a crossing in terms of its opposite plus a contribution from parallel strands, that is

$$V(\text{crossing}) = \tau^2 V(\text{opposite crossing}) + (\tau^{\frac{3}{2}} - \tau^{\frac{1}{2}}) V(\text{parallel strands}) . \tag{9}$$

By applying this relation to the encircled crossing of each trefoil knot we can transform the top diagrams of Figure 3 into their relative decompositions given by a writhe and a Hopf link (bottom diagrams). With reference to the left-handed trefoil of Figure 3a, we have a writhe  $\gamma_-$  and a Hopf link  $H_+$ . For the writhe, by using (8), we have (see Figure 1a)

$$\tau^{-1} V(\gamma_+) - \tau V(\gamma_-) = (\tau^{\frac{1}{2}} - \tau^{-\frac{1}{2}}) V(\mathbf{l}_{cc}) , \tag{10}$$

that gives

$$V(\mathbf{l}_{cc}) = -\tau^{-\frac{1}{2}} - \tau^{\frac{1}{2}} , \tag{11}$$

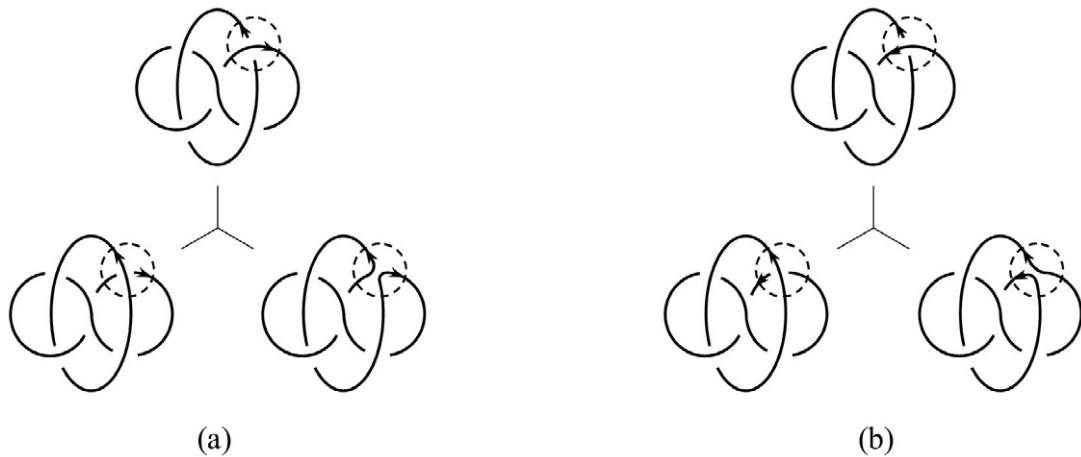


Fig. 4. Reduction schemes for Whitehead links  $W_+$  and  $W_-$ . (a) Top: Whitehead link  $W_+$  with crossing  $+1$ ; bottom: Hopf link  $H_-$  and left-handed trefoil knot  $T^L$ . (b) Top: Whitehead link  $W_-$  with crossing  $-1$ ; bottom: Hopf link  $H_+$ , and figure-of-eight knot  $F^8$ .

because  $V(\gamma_+) = V(\gamma_-) = V(\bigcirc) = 1$ . Note that the orientation of any number of disjoint rings has no effect on the polynomial. As regards to the Hopf link  $H_+$ , with reference to Figure 1b, we have

$$\tau^{-1}V(H_+) - \tau V(l_{cc}) = (\tau^{\frac{1}{2}} - \tau^{-\frac{1}{2}})V(\gamma_+), \tag{12}$$

that gives

$$V(H_+) = -\tau^{\frac{1}{2}} - \tau^{\frac{5}{2}}. \tag{13}$$

Similarly for the Hopf link  $H_-$  of the right-handed trefoil knot of Figure 3b. With reference to Figure 1c, we have

$$\tau^{-1}V(l_{cc}) - \tau V(H_-) = (\tau^{\frac{1}{2}} - \tau^{-\frac{1}{2}})V(\gamma_-), \tag{14}$$

that gives

$$V(H_-) = -\tau^{-\frac{1}{2}} - \tau^{-\frac{5}{2}}. \tag{15}$$

By combining the above results we have: for the left-handed trefoil knot  $T^L$

$$\tau^{-1}V(\gamma_-) - \tau V(T^L) = (\tau^{\frac{1}{2}} - \tau^{-\frac{1}{2}})V(H_-), \tag{16}$$

that, by using (15), gives

$$V(T^L) = \tau^{-1} + \tau^{-3} - \tau^{-4}. \tag{17}$$

For the right-handed trefoil knot  $T^R$ , we have

$$\tau^{-1}V(T^R) - \tau V(\gamma_+) = (\tau^{\frac{1}{2}} - \tau^{-\frac{1}{2}})V(H_+). \tag{18}$$

Thus, by using (13), we have

$$V(T^R) = \tau + \tau^3 - \tau^4. \tag{19}$$

By comparing (17) with (19) we see that the two mirror knots have different polynomials.

3.2. Whitehead link

A second example is provided by the Whitehead link  $\mathbf{W}$  (see Figure 4). With reference to the bottom diagrams of Figure 4, by applying the skein relation (8) to the Whitehead link  $\mathbf{W}_+$  (of crossing  $+1$ ), we have the relation

$$\tau^{-1}V(\mathbf{W}_+) - \tau V(\mathbf{H}_-) = (\tau^{\frac{1}{2}} - \tau^{-\frac{1}{2}})V(\mathbf{T}^L), \tag{20}$$

and application of (8) to the Whitehead link  $\mathbf{W}_-$  gives

$$\tau^{-1}V(\mathbf{H}_+) - \tau V(\mathbf{W}_-) = (\tau^{\frac{1}{2}} - \tau^{-\frac{1}{2}})V(\mathbf{F}^8), \tag{21}$$

where  $\mathbf{F}^8$  denotes the figure-of-eight knot shown at the bottom of Figure 4b. This latter can be further reduced according to the diagrams of Figure 5. By applying (8) to the unknot with two writhe  $\gamma_-$ , denoted by  $\gamma_=$ , and to the Hopf link with writhe  $\gamma_+$ , denoted by  $\mathbf{H}_+^+$ , we have

$$\tau^{-1}V(\mathbf{F}^8) - \tau V(\gamma_=) = (\tau^{\frac{1}{2}} - \tau^{-\frac{1}{2}})V(\mathbf{H}_+^+). \tag{22}$$

Now, since  $V(\gamma_=) = 1$  and  $V(\mathbf{H}_+^+) = V(\mathbf{H}_-) = -\tau^{-\frac{1}{2}} - \tau^{-\frac{5}{2}}$ , we have

$$V(\mathbf{F}^8) = \tau^{-2} - \tau^{-1} + 1 - \tau + \tau^2. \tag{23}$$

As can be easily verified, the mirror image of the figure-of-eight knot of Figure 5 has the same Jones polynomial of equation (23).

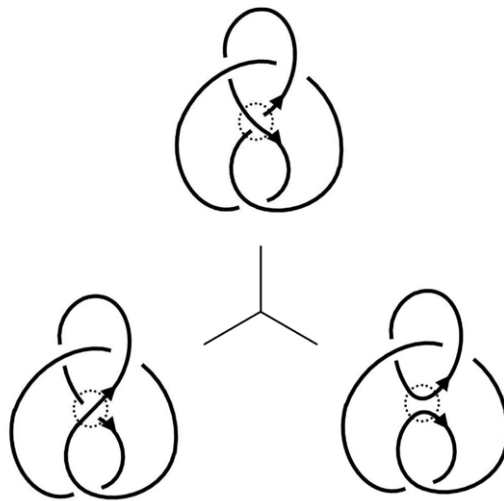


Fig. 5. Reduction scheme for (top) the figure-of-eight knot  $\mathbf{F}^8$ ; bottom: the unknot with two writhe  $\gamma_-$ , denoted by  $\gamma_=$  (left), and a Hopf link with writhe  $\gamma_+$ , denoted by  $\mathbf{H}_+^+$ .

Hence, by substituting (15) and (17) into (20), we have the Jones polynomial for  $\mathbf{W}_+$ . By similar, straightforward computation we obtain also the Jones polynomial for  $\mathbf{W}_-$ . The two polynomials coincide, that is  $V(\mathbf{W}_+) = V(\mathbf{W}_-) = V(\mathbf{W})$ , given by

$$V(\mathbf{W}) = \tau^{-\frac{7}{2}} - 2\tau^{-\frac{5}{2}} + \tau^{-\frac{3}{2}} - 2\tau^{-\frac{1}{2}} + \tau^{\frac{1}{2}} - \tau^{\frac{3}{2}}, \tag{24}$$

indicating that the two knots are actually the same knot type.

#### 4. Knot polynomials as means to quantify structural complexity of vortex flows

For practical applications it is useful to go back to the original position, i.e.  $\tau \rightarrow t^H \rightarrow e^H$ , by referring to  $e^H$  rather than  $\tau$ . By (6) we can write knot polynomials for a vortex tangle of filaments as function of topology and  $H(\gamma_+)$ , where the latter can be interpreted as a reference mean-field helicity of the physical system. Indeed, by (4), we can think of  $H(\gamma_+)$  as a gauge for a mean writhe (or twist) helicity of the background flow. Since in any case this contributes in terms of an average circulation  $\kappa$ , we can re-interpret the Jones polynomial as

$$V_\tau(\mathcal{K}) \rightarrow V_t(\mathcal{K}, \kappa) \rightarrow V(\mathcal{K}, \kappa) . \tag{25}$$

For example, in the case of a homogeneous, isotropic tangle of superfluid filaments, all vortices have same circulation  $\kappa$ ; we can set

$$\bar{\lambda} = \langle \lambda \rangle = \frac{1}{2} , \quad \langle H(\gamma_+) \rangle = \frac{\kappa^2}{2} , \tag{26}$$

where angular brackets denote average values. Hence,

$$\tau = t^{-\kappa^2} \rightarrow e^{-\kappa^2} . \tag{27}$$

In this case, we would have

$$V(\bigcirc) = V(\gamma_+) = V(\gamma_-) = V(\gamma_\pm) = 1 , \tag{28}$$

$$V(\mathbf{l}_{cc}) = -e^{\frac{\kappa^2}{2}} (1 + e^{-\kappa^2}) , \tag{29}$$

$$V(\mathbf{l}_{c,\dots,c}) = [-e^{\frac{\kappa^2}{2}} (1 + e^{-\kappa^2})]^{n-1} , \quad (n \text{ rings}) , \tag{30}$$

$$V(\mathbf{H}_+) = -e^{-\frac{\kappa^2}{2}} (1 + e^{-2\kappa^2}) , \tag{31}$$

$$V(\mathbf{H}_-) = -e^{\frac{\kappa^2}{2}} (1 + e^{2\kappa^2}) , \tag{32}$$

$$V(\mathbf{T}^L) = e^{\kappa^2} + e^{3\kappa^2} - e^{4\kappa^2} , \tag{33}$$

$$V(\mathbf{T}^R) = e^{-\kappa^2} + e^{-3\kappa^2} - e^{-4\kappa^2} . \tag{34}$$

$$V(\mathbf{F}^8) = e^{2\kappa^2} - e^{\kappa^2} + 1 - e^{-\kappa^2} + e^{-2\kappa^2} , \tag{35}$$

$$V(\mathbf{W}) = e^{-\frac{3}{2}\kappa^2} \left( -1 + e^{\kappa^2} - 2e^{2\kappa^2} + e^{3\kappa^2} - 2e^{4\kappa^2} + e^{5\kappa^2} \right) . \tag{36}$$

The examples above are of course elementary. Knot polynomials for more complex systems can be straightforwardly computed by implementing the skein relations in a software code. Knot polynomial invariants can thus provide a powerful tool to acquire new information. Numerical implementation of diagram analysis of complex tangles of filaments has proven useful in the analysis of energy-complexity relations [3]; similarly, implementation of reduction techniques to compute Jones polynomials of knots and links should be amenable to numerical coding without particular difficulty. Even in presence of vortex reconnection, where there is a continuous change of topology, computation of the Jones polynomials would give real-time information on the structural evolution of vorticity. Information obtained by the numerical implementation of these new concepts can therefore provide new means to quantify aspects of structural complexity [9, 10], especially in relation to energy transfers in highly turbulent flows. Moreover, new interesting problems associated with specific physical assumptions on the crossing state splitting decompositions may arise.

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