

# Tackling Fluid Tangles Complexity by Knot Polynomials

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**Abstract.** In this paper we present an application of a new technique, based on recent work done by Liu & Ricca (2012), to quantify structural complexity by means of topological methods. These rely on the derivation of the Jones polynomial from the helicity of ideal fluid flows. The techniques discussed here can be extended and applied to real fluid flows subject to continuous topological restructuring.

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## FLUID TANGLES AND HELICITY

A common feature in the development of turbulence, both in classical and quantum context, is the emergence of complex tangles of vortex structures (Kerr, 2005; Baggaley, 2012), disorderly distributed in the fluid domain (see Figure 1a). If the fluid is electrically charged or highly magnetized, as in the plasmas of stellar atmospheres or in confined devices, these tangles are made up of an intricate network of flux tubes in space. For such complex tangles of filaments helicity plays a fundamental role, since it is an invariant of the Euler and ideal magnetohydrodynamics (MHD) equations, and a robust quantity of the dissipative Navier–Stokes equations and resistive MHD flows.

Moreover, its topological interpretation in terms of Gauss linking number, originally given by Moffatt (1969) and then extended by Ricca & Moffatt (1992) in the context of vortex dynamics (kinetic helicity) and MHD (magnetic helicity, see Moffatt & Ricca, 1992), has become the standard reference in topological dynamics. Kinetic helicity is defined by

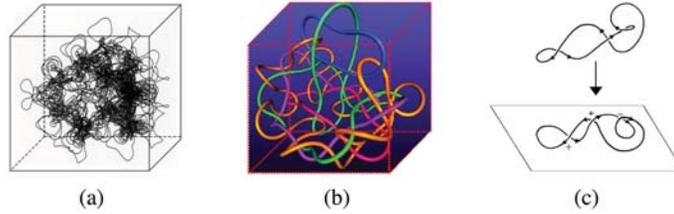
$$H \equiv \int_{\Omega} \mathbf{u} \cdot \boldsymbol{\omega} d^3 \mathbf{x}, \quad (1)$$

where  $\mathbf{u}$  is the velocity field, defined on an unbounded domain of  $\mathbb{R}^3$ ,  $\boldsymbol{\omega} = \nabla \times \mathbf{u}$  is vorticity, defined on a sub-domain  $\Omega$ , and  $\mathbf{x}$  a position vector of  $\mathbb{R}^3$ . For simplicity we assume  $\nabla \cdot \mathbf{u} = 0$  everywhere, and we request  $\boldsymbol{\omega} \cdot \hat{\mathbf{n}} = 0$  on  $\partial\Omega$ , where  $\hat{\mathbf{n}}$  is orthogonal to  $\partial\Omega$ , with  $\nabla \cdot \boldsymbol{\omega} = 0$ . Magnetic helicity, on the other hand, is simply obtained by replacing  $\mathbf{u} \cdot \boldsymbol{\omega}$  with  $\mathbf{A} \cdot \mathbf{B}$ , where  $\mathbf{B} = \nabla \times \mathbf{A}$  is the magnetic field in  $\Omega$  and  $\mathbf{A}$  is its vector potential, subject to the Coulomb gauge condition  $\nabla \cdot \mathbf{A} = 0$ . Indeed any dot product of a field and its curl (such as  $\mathbf{B}$  and the electric current density  $\mathbf{J} = \nabla \times \mathbf{B}$ ) can be seen as the kernel of a helicity integral. A remarkable progress has been recently made by Liu & Ricca (2012), who proved that knot polynomials based on helicity provides a new, powerful means to quantify the topological complexity of a tangle of fluid knots. Here we present these results, discussing their physical interpretation for new possible implementations in visimetrics and advanced diagnostics in direct numerical simulation of complex flow fields.

For simplicity let us consider a tangle of  $N$  such fluid filaments associated with the field  $\mathbf{A}$  (velocity field, magnetic field, vector potential, etc.) in ideal conditions. Then,  $\Omega$  may be thought of as a disjoint union of  $N$  tubular knots (see Figure 1b), each being centred on a smooth, oriented curve  $\bar{\gamma}_i$  ( $i = 1, \dots, N$ ) in  $\mathbb{R}^3$ . We may refer to this collection of physical knots as an oriented knot or link  $\mathcal{K}$  in  $\mathbb{R}^3$ . In this case the helicity  $H = H(\mathcal{K})$  can be reduced to

$$H = \sum_i \kappa_i \oint_{\bar{\gamma}_i} \mathbf{A} \cdot d\mathbf{l}, \quad (2)$$

where  $\kappa_i$  is the flux of  $\nabla \times \mathbf{A}$  in the  $i$ -th tubular knot, and  $d\mathbf{l}$  an elementary arc-length of  $\bar{\gamma}_i$  in the tangent direction to  $\bar{\gamma}_i$ . Note that the flux is a topological charge of the ambient fluid and is an invariant of fluid flows. For simplicity we



**FIGURE 1.** (a) A superfluid vortex tangle obtained by direct numerical simulation from the interaction of an initial seed vorticity. (b) A disjoint union of tubular knots. (c) Indented projections of oriented knots and links in space provide oriented diagrams with over-crossings and under-crossings, denoted by + or - signs (according to standard convention) in the diagram.

may also assume the all tubular knots have same flux, that is  $\kappa_i = \kappa$  for any  $i = 1, \dots, N$  (indeed, for simplicity, we can set  $\kappa = 1$ ).

### KNOT POLYNOMIALS FROM THE HELICITY OF FLUID FLOWS

Topological information of knots and links in  $\mathbb{R}^3$  can be extracted by diagram analysis performed on indented projections (Kauffman, 1987). An "indented" projected diagram of a knot or link in space to a plane is obtained by keeping track of the over-crossings and under-crossings of the oriented strands along the direction of projection chosen (see Figure 1c). This is done by assigning a + or a - sign to each apparent crossing in the projected diagram, according to standard convention. By indenting the apparent crossings, topological information is therefore preserved and one can then work on the indented diagram representation of the oriented knot or link. Note that topological information can be retrieved regardless of the direction of projection chosen. In this context a powerful topological invariant of oriented knots and links  $\mathcal{K}$  in  $\mathbb{R}^3$  is provided by the Jones polynomial  $V = V(\mathcal{K})$ . This is defined by the skein relations, presented here in this alternative form

$$V(\bigcirc) = V(\vec{\gamma}_+) = V(\vec{\gamma}_-) = 1 \quad (3)$$

$$V\left(\begin{array}{c} \nearrow \\ \searrow \end{array}\right) = \tau^2 V\left(\begin{array}{c} \nearrow \\ \nearrow \end{array}\right) + (\tau^{\frac{3}{2}} - \tau^{\frac{1}{2}}) V\left(\begin{array}{c} \nearrow \\ \leftarrow \end{array}\right). \quad (4)$$

Here  $\tau$  is a dummy variable and it has no physical meaning. Equations (3) simply state that the polynomial of the standard oriented circle is equal to that of the positive or negative writhe (since they are topologically equivalent to the circle), denoted respectively by  $\vec{\gamma}_+$  and  $\vec{\gamma}_-$  (see Figure 2a). Equation (4) states that the polynomial of a knot associated with a positive crossing (on the left-hand-side) is related to that given by replacing that crossing with a negative crossing and then that crossing with a non-crossing of parallel strands. What matters here is that the polynomial expression of  $V$  is a topological invariant of  $\mathcal{K}$ . Thus, for any given knot or link, by recursive application of the skein relations one can compute the Jones polynomial of that knot or link (up to smooth, tame representations). See the examples of Figure 2b and 2c for an elementary illustration.

Recently, Liu & Ricca (2012) have proven the following result:

**Theorem 1** Let  $\vec{\mathcal{K}}$  denote a fluid knot or an  $N$ -component link. If the helicity of  $\vec{\mathcal{K}}$  is  $H = H(\vec{\mathcal{K}})$ , then

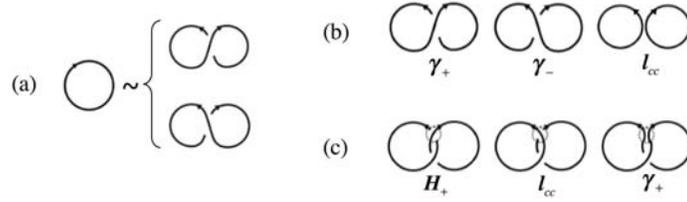
$$e^{H(\vec{\mathcal{K}})} = e^{\oint_{\vec{\mathcal{K}}} \mathbf{A} \cdot d\mathbf{l}}, \quad (5)$$

appropriately re-scaled, satisfies the skein relations of the Jones polynomial  $V = V(\vec{\mathcal{K}})$ .

Detailed proof of the derivation of this result, as well as explicit computations of the Jones polynomial for the left- and right-handed trefoil knots and the Whitehead link, are given in that paper. In general we have

$$\tau = e^{-4\lambda H(\vec{\gamma}_+)}, \quad (0 \leq \lambda \leq 1), \quad (6)$$

where  $\lambda$  takes into account of the uncertainty associated with the writhe value of  $\vec{\gamma}_+$  and  $H(\vec{\gamma}_+)$  is the helicity of  $\vec{\gamma}_+$ . The result above, given by (1) and (6), establishes a new, fundamental connection between topological measures of structural complexity of filament tangles and physical information. This will be discussed further in the next section.



**FIGURE 2.** (a) The oriented circle is topologically equivalent to both, the positive writhe  $\vec{\gamma}_+$  (top diagram) and the negative writhe  $\vec{\gamma}_-$ , thus, according to (3), all of them have same Jones polynomial  $V = 1$ . (b) By applying the skein relations (3) and (4) to the positive writhe  $\vec{\gamma}_+$ , we obtain  $V(\vec{l}_{cc}) = -\tau^{-\frac{1}{2}} - \tau^{\frac{1}{2}}$ . (c) The Jones polynomial of the positive Hopf link  $\vec{H}_+$  is given by  $V(\vec{H}_+) = -\tau^{\frac{1}{2}} - \tau^{\frac{5}{2}}$ .

## PHYSICAL INFORMATION FROM TOPOLOGICAL COMPLEXITY

For the sake of illustration, let us consider the case of a homogeneous tangle of filaments of equal circulation  $\kappa$ , so that we can set

$$\lambda = \langle \lambda \rangle = \frac{1}{2}, \quad \langle H(\vec{\gamma}_+) \rangle = \frac{\kappa^2}{2}, \quad (7)$$

where the two values above represents the mean values associated with the uncertainty of writhing configurations and actual amount of background (twist) helicity present in the flow (a reference or gauge value of the fluid helicity). With these simple assumptions, we have

$$\tau = e^{-\kappa^2}. \quad (8)$$

Thus, we can relate the Jones polynomial to physical information by writing

$$V(\vec{\mathcal{K}}(\tau)) \rightarrow V(\vec{\mathcal{K}}(\kappa)) = f(\vec{\mathcal{K}}; \kappa), \quad (9)$$

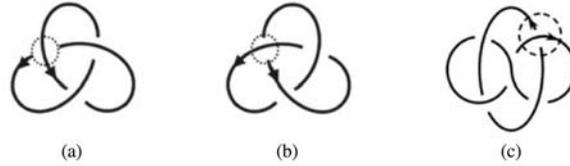
where  $f(\vec{\mathcal{K}}; \kappa)$  is indeed a function of topology, through the Jones polynomial, and physics, through the flux.

### Computation of elementary cases

As mentioned in the previous section, explicit computation of the Jones polynomial for a few simple cases given by the left- and right-handed trefoil knots and the Whitehead link is done for illustration in Liu & Ricca (2012). Let us re-interpret these polynomials by making use of (8). We have:

- Unknot, positive or negative writhe:  $V(\bigcirc) = V(\vec{\gamma}_+) = V(\vec{\gamma}_-) = 1$  ;
- Two unlinked, unknotted circles:  $V(\vec{l}_{cc}) = -e^{\frac{\kappa^2}{2}} - e^{-\frac{\kappa^2}{2}}$  ;
- Positive Hopf link:  $V(\vec{H}_+) = -e^{-\frac{\kappa^2}{2}} - e^{-\frac{5\kappa^2}{2}}$  ;
- Negative Hopf link:  $V(\vec{H}_-) = -e^{\frac{\kappa^2}{2}} - e^{\frac{5\kappa^2}{2}}$  ;
- Left-hand trefoil knot:  $V(\vec{T}^L) = e^{\kappa^2} + e^{3\kappa^2} - e^{4\kappa^2}$  ;
- Right-hand trefoil knot:  $V(\vec{T}^R) = e^{-\kappa^2} + e^{-3\kappa^2} - e^{-4\kappa^2}$  ;
- Whitehead link (positive or negative):  $V(\vec{W}) = -e^{-\frac{3\kappa^2}{2}} \left( 1 - e^{\kappa^2} + 2e^{2\kappa^2} - e^{3\kappa^2} + 2e^{4\kappa^2} - e^{5\kappa^2} \right)$  .

This list provides an example of how topological complexity and physical information can be combined together to give measures that are invariant under ideal fluid evolution.



**FIGURE 3.** Three examples of oriented knots and links: (a) left-handed trefoil knot (b) right-handed trefoil knot and (c) Whitehead link. Encircled crossing sites can be used as a starting point for recursive application of the skein relations to derive the Jones polynomial.

### Application to real fluid flows

Realistic fluid tangles are not only far more complex than the simple cases illustrated in the previous section, but they are subject to continuous rearrangement due to reconnection processes and dissipative effects. The techniques discussed here can nevertheless be profitably extended and applied to these dynamical systems. In the more general case of adaptive topology, numerical implementation of real-time analysis of the concepts discussed above seems readily feasible and it could be tested straightaway by using fluid flow databases to estimate helicity and energy transfers. This novel approach will open up new directions of work to explore and quantify aspects of structural complexity (Ricca, 2009), central in modern fluid flow visualization and analysis.

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