THE HELICITY OF A KNOTTED VORTEX FILAMENT

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ABSTRACT. The helicity $\mathcal{H}$ associated with a knotted vortex filament is considered. The filament is first constructed starting from a circular tube, in three stages involving injection of (integer) twist, deformation and switching of crossings. This produces a vortex tube in the form of an arbitrary knot K; each vortex line in the tube is a (trivial) satellite of K, and the linking number of any pair of vortex lines in the tube is the same integer n. It is shown that in these circumstances the helicity is given by $\mathcal{H} = n\kappa^2$ where κ is the circulation associated with the tube. This result is discussed in relation to earlier works, in particular the work of Călugăreanu (1959, 1961) which establishes that, for a twisted ribbon with axis $\mathcal{C}$ the number n is the sum of three ingredients:

$$\frac{\mathcal{H}}{\kappa^2} = n = W(\mathcal{C}) + \mathcal{T}(\mathcal{C}) + \frac{1}{2\pi}[\Delta\Theta]_{\mathcal{C}}$$

where $W(\mathcal{C})$ is the writhing number and $\mathcal{T}(\mathcal{C})$ is the total torsion. The quantity $[\Delta\Theta]_{\mathcal{C}}$ represents the net angle of rotation of the spanwise vector on the ribbon relative to the Frenet triad in one passage round $\mathcal{C}$. Both $\mathcal{T}(\mathcal{C})$ and $[\Delta\Theta]_{\mathcal{C}}$ are discontinuous in deformations that take $\mathcal{C}$ through an inflexion point. The generic behaviour in such passage through an inflexion point is analysed and clarified in §6.

1. Introduction

Let $\mathbf{u}(\mathbf{x}, t)$ be the velocity field in an inviscid incompressible fluid, evolving under the Euler equations, and let $\omega(\mathbf{x}, t) = \nabla \times \mathbf{u}$ be the corresponding vorticity field. Let $\mathcal{S}$ be any closed orientable surface moving with the fluid on which $\omega \cdot \mathbf{n} = 0$. Then it is well-known (Moffatt, 1969) that the helicity integral

$$\mathcal{H} = \int_{\mathcal{V}} \mathbf{u} \cdot \omega \, d\mathcal{V} \tag{1}$$

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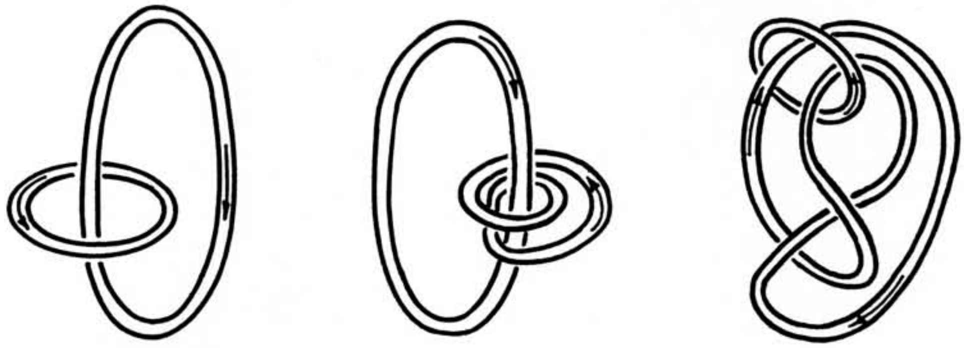

a) $N = +1$ b) $N = -2$ c) $N = 0$

Fig. 1. Linking of oriented vortex tubes.

where $\mathcal{V}$ is the volume inside $\mathcal{S}$, is an invariant under this Euler evolution, this invariance being associated with the fact that the vortex lines are frozen in the fluid and the topology of the vorticity field is therefore conserved.

The topological interpretation is most transparent for the simple situation in which $\omega \equiv 0$ except in two linked vortex filaments of vanishingly small cross-sections and of circulations κ_1, κ_2; then, provided each vortex tube is unknotted and the vorticity field has no internal twist within each tube, it is easily shown that

$$\mathcal{H} = 2N\kappa_1\kappa_2 \tag{2}$$

where N is the Gauss (linking) number of the axes C_1, C_2 of the tubes, positive or negative according as the orientation of the linkage is right-handed or left-handed (for examples, see Fig. 1).

This interpretation has been extended by Arnol'd (1974) to situations in which linked vortex lines are not closed curves but wind around each other infinitely often. The integral (1) is still invariant in this situation and has been described by Arnol'd as the "asymptotic Hopf invariant".

For a single knotted vortex filament, the situation is not so simple. If the axis of the tube is in the form of a knot K, then each vortex line is (if closed) a satellite of K, and the helicity invariant may be expected to bear the imprint of K in the limit as the tube cross-section shrinks to zero. However there is now an unavoidable twist of the field ω within the tube, partly associated with torsion of the axis C of the tube, and evaluation of $\mathcal{H}$ presents consequential difficulties.

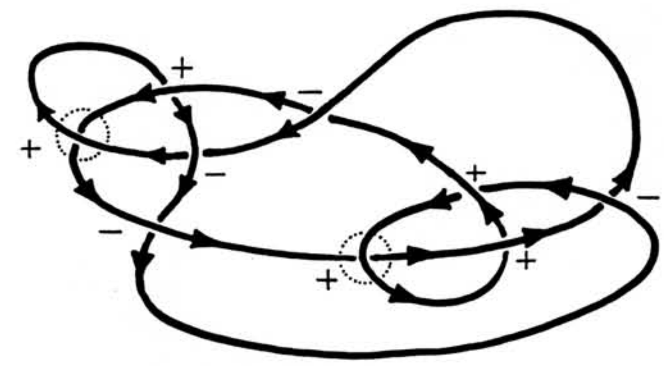

Fig. 2. Alternating knot (overcrossings alternate with undercrossings) with 9 crossings, 5 positive and 4 negative (at a positive crossing, the overstrand must be rotated anticlockwise to come into coincidence with the understrand with arrows pointing the same way). Switching of two positive crossings converts this to the unknot.

It has been conjectured (Moffatt, 1981) that $\mathcal{H}$ must be asymptotically identifiable with the invariant of Călugăreanu (1959, 1961) and this is indeed implicit in later works (*e.g.* Berger & Field, 1984) in which the helicity of a knotted vortex tube (or equivalently magnetic flux tube) is expressed as the sum of writhe and twist components. However a direct evaluation of $\mathcal{H}$ for a knotted vortex filament has never been given. We aim to provide this in the present contribution.

2. Construction of a knotted vortex tube of prescribed helicity

We recall first the construction of a knotted vortex tube described by Moffatt (1990). Let K be an arbitrary (tame) knot, and suppose that we view it in standard plane projection with a finite number of crossings, each of which is either positive or negative (for an example, see Fig. 2). By a finite number of crossing "switches", K may be converted to the unknot, which may be continuously deformed to a circle $\mathcal{C}_0$. By reversing these steps, $\mathcal{C}_0$ may be reconverted to K.

Now let T_0 be a tubular neighbourhood of $\mathcal{C}_0$, and let ω_0 be a vorticity field in T_0, uniform over each (small) cross-section of T_0, each vortex line being a circle parallel to $\mathcal{C}_0$. Let κ be the circulation of the vortex tube. The helicity of ω_0 is zero. We may inject helicity $\mathcal{H}_0 = h\kappa^2$ into this vorticity field by *Dehn surgery*: cut the tube at some section, twist through an angle $2\pi h$, and reconnect (Fig. 3). If h is an integer n_0 (as we shall suppose), then each

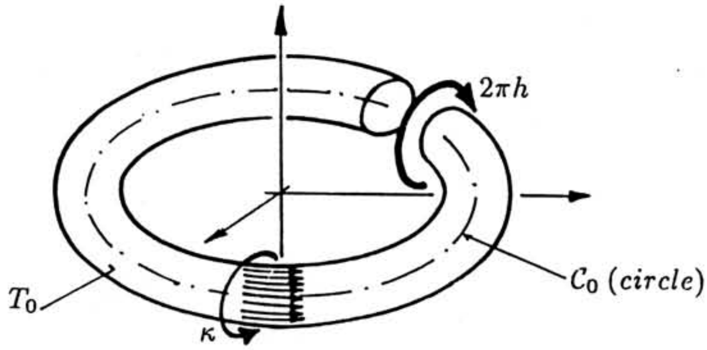

Fig. 3. Dehn surgery (cut, twist, reconnect) on a circular vortex tube.

ω line is a closed curve which closes on itself after one passage around T_0 (in knot terminology, it has winding number 1). Now convert C_0 (carrying T_0 with it) to the knot K by the steps indicated above. $\mathcal{H}_0$ is conserved during deformation but changes by an amount $\pm 2\kappa^2$ with each switch creating a positive or negative crossing (Fig. 4). Hence the helicity of the knotted tube becomes

$$\mathcal{H} = (n_0 + 2N_+ - 2N_-)\kappa^2 = n\kappa^2, \tag{3}$$

say, where $N_\pm$ are (respectively) the numbers of positive and negative switches used to create K. Thus, by this construction, $\mathcal{H}/\kappa^2$ is an integer.

Note the special choice $n_0 = -2(N_+ - N_-)$ which makes $\mathcal{H} = 0$. In this situation, each ω-line is still a replica of K so that the topology of the ω-field is decidedly non-trivial. However the linking number of each pair of ω-lines is zero (this does not necessarily mean that they are unlinked! — see the example of Fig. 1c).

3. Helicity and the self-linking number of a framed knot

The number n in (3) is in fact the linking number of any pair of ω-lines in the knotted tube filament. This may be proved as follows.

Let us divide the tube up into m "sub-tubes" each with the same circulation (flux of vorticity) κ/m. Suppose that the linking number of each pair of vortex lines is N; this is then also the linking number of each pair of sub-tubes.

If $\mathcal{H}$ is the total helicity, then the helicity associated with the vorticity in a sub-tube is

$$\mathcal{H}_m = \mathcal{H}/m^2 \tag{4}$$

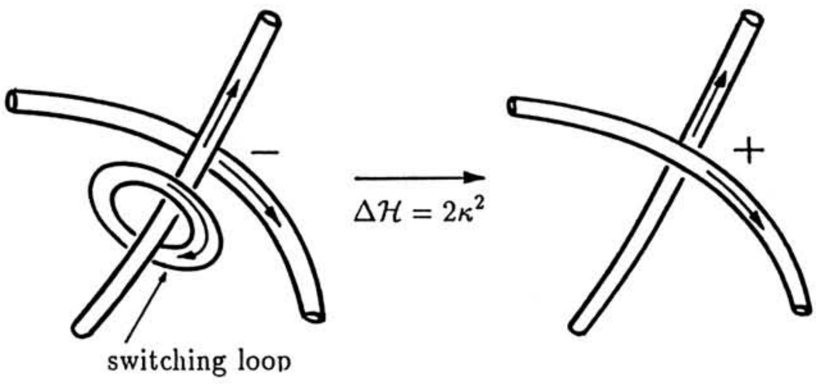

Fig. 4. Switching a negative crossing to become a positive crossing; this is equivalent to the insertion of a "switching loop" which cancels the field of the understrand below the crossing and recreates it above the crossing. This corresponds to increasing the helicity of the knot by $2\kappa^2$. Similarly the reverse switch changes the helicity by $-2\kappa^2$.

(since helicity is a quadratic functional of vorticity). $\mathcal{H}_m$ may be thought of as the "self-helicity" of a sub-tube T_m associated with the linkage of vortex lines within T_m.

The total helicity $\mathcal{H}$ is the sum of these self-helicities plus the sum of the interactive helicities (cf. eq. 2) arising from linkage of flux tubes, *i.e.*

$$\mathcal{H} = m\mathcal{H}_m + 2\sum_{\substack{i,j \\ i\neq j}} N\kappa_i\kappa_j \tag{5}$$

with $\kappa_i = \kappa/m$ $(i = 1, 2, \ldots, m)$. Hence

$$\mathcal{H} = \frac{\mathcal{H}}{m} + 2N\frac{1}{2}m(m-1)\left(\frac{\kappa}{m}\right)^2 \tag{6}$$

i.e.

$$\mathcal{H} = N\kappa^2, \tag{7}$$

a result that is independent of the degree m of subdivision of the tube. Hence, comparing with (3), $N = n$ as asserted.

Any two vortex lines C_1 and C_2 in the tube are the boundaries of a ribbon R_{12} contained within the tube. A frame of reference $(\mathbf{e}_1, \mathbf{n}_{12}, \mathbf{e}_1 \times \mathbf{n}_{12})$ may be constructed on this ribbon, where $\mathbf{e}_1$ is the tangent vector to C_1 (a function of arc-length s_1 on C_1), and $\mathbf{n}_{12}$ is the spanwise vector on R_{12} from C_1 to C_2 (also a function of s_1). Choice of such a frame constitutes a "framing" of

I

II

III

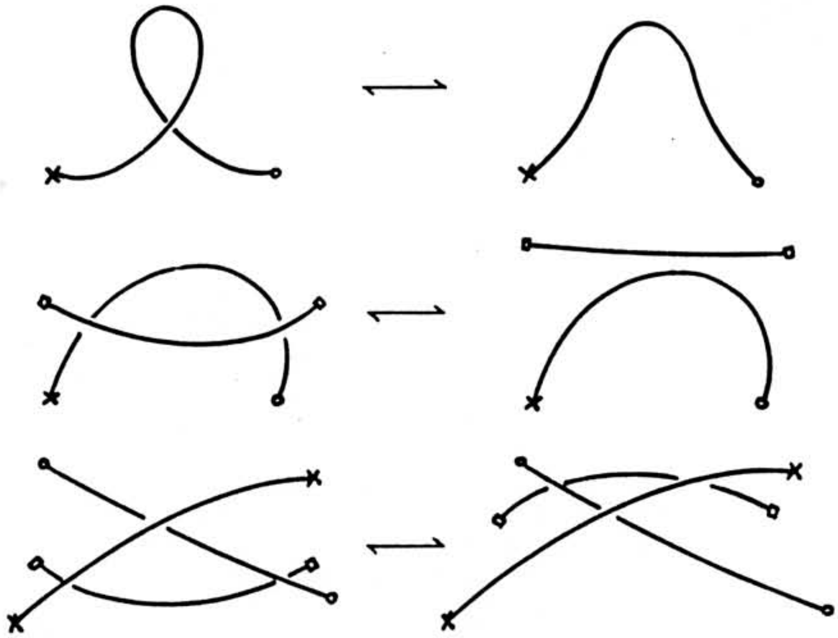

Fig. 5. The three Reidemeister moves (Kauffman, p. 9, 1987).

the knot K. The number n may then be described (Pohl, 1968) as the self-linking number of the framed knot. The self-linking number has no meaning unless a frame is specified.

4. Helicity and Reidemeister's first move

One of the classical results of knot theory is that two knots K and K' are isotopic (*i.e.* one may be continuously deformed into the other) if and only if this can be achieved by a succession of Reidemeister moves (Fig. 5) acting on (any) plane projection of K (or K'). A distinction is made between "ambient isotopy" which allows moves of types I, II and III, and "regular isotopy" which allows only moves of types II and III. The description "ambient isotopic" is synonymous (for knots) with "topologically equivalent" (Kauffman, p. 9, 1987).

If we consider the Reidemeister moves applied to a vortex filament, then we can think of these as being localised deformations of the vorticity field that conserve global helicity. There is no difficulty whatsoever in relation to the moves II and III in this respect. There is however a subtlety in relation to move I, as may be seen from consideration of figure 6 which represents the twisting of a loop through an angle 2π, thus converting a negative crossing

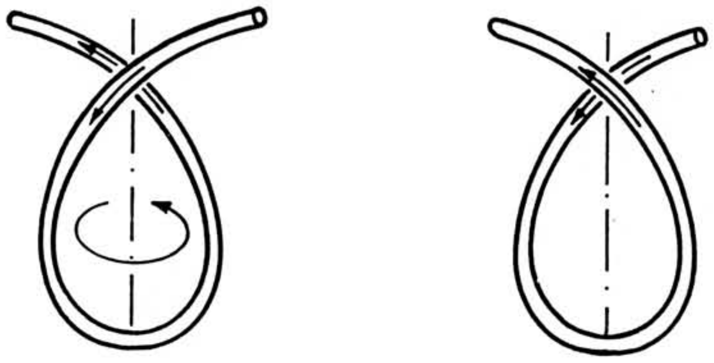

Fig. 6. A right-handed twist of the loop through 2π (equivalent to two type-I Reidemeister moves) converts a negative crossing to a positive crossing.

to a positive crossing. If the same effect were achieved by a crossing switch as described in §2 above, then the helicity would increase by $2\kappa^2$. Since the Reidemeister twisting move conserves helicity, this increase must be compensated by an equal decrease $-2\kappa^2$ arising from the twist of ω-lines within the tube, *i.e.* a combination of switching and Dehn surgery is needed to maintain helicity:

$$2 \times \text{Reidemeister move I} \equiv \text{switch } (- \to +) + \text{Dehn surgery } (-4\pi) \qquad (8)$$

The essence of this equivalence may be appreciated by playing with a belt or a paper ribbon!

5. Helicity and the Călugăreanu invariant

Now let C be the central axis of the vortex tube (itself a vortex line); let s be the arc-length from a point O of C, and suppose that the curvature $c(s)$ is everywhere positive on C (*i.e.* there are no points of inflexion). Let $(\mathbf{e}, \mathbf{n}, \mathbf{b})$ be the orthogonal triad of unit vectors ($\mathbf{e}$ = tangent vector, $\mathbf{n}$ = principal normal, $\mathbf{b}$ = binormal) which satisfy the Frenet-Serret equations

$$\frac{d\mathbf{e}}{ds} = c\mathbf{n}, \qquad \frac{d\mathbf{n}}{ds} = -c\mathbf{e} + \tau\mathbf{b}, \qquad \frac{d\mathbf{b}}{ds} = -\tau\mathbf{n}, \qquad (9)$$

where $\tau(s)$ is the torsion of C.

If $\mathbf{x} = \mathbf{x}(s)$ is a point on C, then a neighbouring curve C' may be defined by $\mathbf{x}' = \mathbf{x}(s) + \varepsilon\mathbf{n}(s)$ where ε is a small positive parameter. The linking number of C and C' is given by Gauss's formula

$$G(C, C') = \frac{1}{4\pi} \oint_C \oint_{C'} \frac{d\mathbf{x} \times d\mathbf{x}' \cdot (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \qquad (10)$$

and this is of course an integer. Călugăreanu (1959) considered the limit of this integral as $\varepsilon \to 0$. One (obvious) contribution in the limit is what is known (Fuller, 1971) as the writhing number of $\mathcal{C}$:

$$W(\mathcal{C}) = \frac{1}{4\pi}\oint_{\mathcal{C}}\oint_{\mathcal{C}} \frac{d\mathbf{x} \times d\mathbf{x}' \cdot (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}. \tag{11}$$

There is however a second contribution to the limit arising from pairs of points $\mathbf{x}, \mathbf{x}'$ such that $|\mathbf{x} - \mathbf{x}'| = O(\varepsilon)$. Călugăreanu showed that this second contribution is given by

$$\mathcal{T}(\mathcal{C}) = \frac{1}{2\pi}\oint_{\mathcal{C}} \tau(s)\, ds \tag{12}$$

i.e. the total torsion divided by 2π, and therefore that

$$\lim_{\varepsilon \to 0} G(\mathcal{C}, \mathcal{C}') = W(\mathcal{C}) + \mathcal{T}(\mathcal{C}) \tag{13}$$

is an invariant under distortions of $\mathcal{C}$ *which do not introduce any inflexion point*. This is a very severe restriction, because as will be shown in the following section it excludes twisting deformations, *i.e.* Reidemeister moves of type I.

6. Generic behaviour associated with inflexion points

At an inflexion point s_c on a curve $\mathbf{x} = \mathbf{x}(s)$ (where s is arc-length), $d\mathbf{e}/ds = d^2\mathbf{x}/ds^2 = 0$, so that near $s = s_c$,

$$\mathbf{e}(s) = \mathbf{e}_c + \frac{1}{2}(s - s_c)^2 \mathbf{e}_c'' + \dots \tag{14}$$

and

$$\mathbf{x}(s) = \mathbf{x}_c + (s - s_c)\mathbf{e}_c + \frac{1}{6}(s - s_c)^3 \mathbf{e}_c'' + \dots \tag{15}$$

Moreover $\mathbf{e}_c''$ is perpendicular to $\mathbf{e}_c$ since

$$(\mathbf{e}'' \cdot \mathbf{e})_{s=s_c} = \frac{d^2}{ds^2}(\mathbf{e}^2)\bigg|_{s=s_c} = 0. \tag{16}$$

Choosing origin at the inflexion point ($\mathbf{x}_c = 0$, $s_c = 0$), and axes $Oxyz$ with Ox parallel to $\mathbf{e}_c$ and Oz parallel to $\mathbf{e}_c''$, the form of the curve near the inflexion is

$$\mathbf{x}(s) = (s,\, 0,\, \alpha s^3) \tag{17}$$

where $\alpha = \frac{1}{6}|\mathbf{e}_c''|$, *i.e.* $y = 0$, $z = \alpha x^3$. Without loss of generality we may take $\alpha = 1$. We consider a time-dependent twisted cubic curve

$$\mathcal{C}: \quad \mathbf{x}(s,t) = \left(s - \frac{2}{3}t^2s^3,\, ts^2,\, s^3\right) \tag{18}$$

which passes through the plane inflexional configuration (17) at time $t = 0$ (Fig. 7). We shall suppose that $|t|$ and $|s|$ are small, and we calculate the torsion $\tau(s,t)$ to leading order near $t = s = 0$. First note that

$$\mathbf{e}(s,t) = \frac{\partial \mathbf{x}}{\partial s} = (1 - 2t^2s^2,\, 2ts,\, 3s^2) \tag{19}$$

and that

$$|\mathbf{e}| = 1 + O(s^4) \tag{20}$$

so that, neglecting terms of order s^4, $\mathbf{e}(s,t)$ is indeed the unit tangent vector near $s = 0$.

We now have

$$\frac{\partial \mathbf{e}}{\partial s} = (-4t^2s,\, 2t,\, 6s) \approx (0,\, 2t,\, 6s) \tag{21}$$

so that the curvature is

$$c(s,t) = \left|\frac{\partial \mathbf{e}}{\partial s}\right| \approx 2(t^2 + 9s^2)^{\frac{1}{2}} \tag{22}$$

near $t = s = 0$. As expected, c vanishes at $s = 0$, $t = 0$, but there is no inflexion point when $|t| \neq 0$; thus $\mathcal{C}(t)$ contains an inflexion point at $s = 0$ at the single instant $t = 0$. The principal normal is

$$\mathbf{n}(s,t) = \frac{1}{c}\frac{\partial \mathbf{e}}{\partial s} = (t^2 + 9s^2)^{-\frac{1}{2}}(0,\, t,\, 3s) \tag{23}$$

and the binormal is then, to leading order,

$$\mathbf{b}(s,t) = \mathbf{e} \times \mathbf{n} = (t^2 + 9s^2)^{-\frac{1}{2}}(0,\, -3s,\, t). \tag{24}$$

From the third Frenet-Serret equation $\partial \mathbf{b}/\partial s = -\tau \mathbf{n}$, we now easily find that to leading order near $t = s = 0$,

$$\tau(s,t) = \frac{3t}{t^2 + 9s^2} \tag{25}$$

This result (see Fig. 8) reveals the nature of the singularity of τ at $t = s = 0$; indeed the total torsion between $s = -s_0$ and $s = +s_0$ for $t \neq 0$ is

$$\int_{-s_0}^{s_0} \tau(s,t)\, ds = 2\int_0^{s_0} \frac{3t}{t^2 + 9s^2}\, ds = 2\tan^{-1}\left(\frac{3s_0}{t}\right). \tag{26}$$

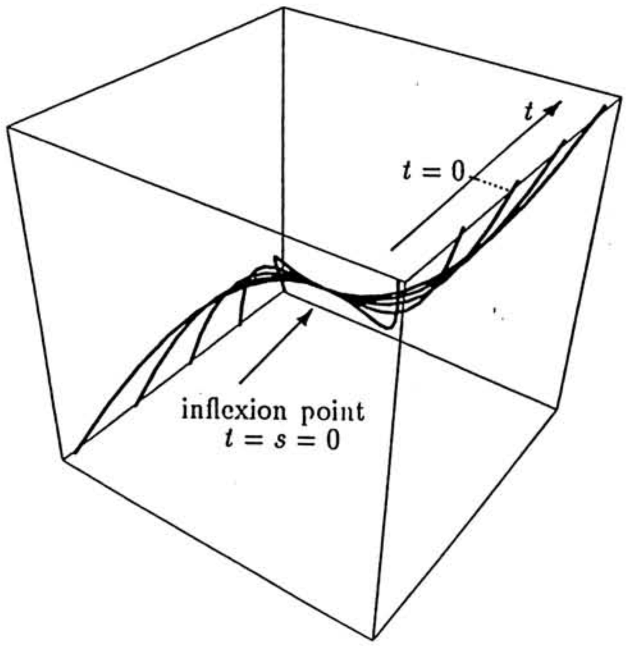

Fig. 7. The twisted cubic (18) for $-1 < s < 1$ and for various values of t. The curve contains an inflexion point at $s = 0$ when $t = 0$.

This jumps from $-\pi$ to $+\pi$ as t increases through zero, irrespective of the value of s_0.

This type of behaviour was recognised by Călugăreanu (1961) who devised a particular example of a *closed* curve deforming through an inflexional configuration. However, Călugăreanu misleadingly refers to "a discontinuity of τ" at the point of inflexion[1], whereas in fact it is the integral with respect to arc-length of $\tau(s,t)$ that is dicontinuous (by an amount 2π) at $t = 0$. This discontinuity is just the amount expected for a Reidemeister move I, by the argument of §4.

Since $W(C)$ is continuous and $\mathcal{T}(C)$ is discontinuous for such distortions, the sum is no longer invariant when distortions through an inflexion point take place. However, the situation is rectified by framing the curve as in §3 and by including the total angle of twist $[\Delta\Theta]_C$ of the spanwise vector from C to C' relative to the Frenet vectors $(\mathbf{n}, \mathbf{b})$ in one passage round C. The "modified" Călugăreanu invariant is then

$$\frac{\mathcal{H}}{\kappa^2} = W(C) + \mathcal{T}(C) + \frac{1}{2\pi}[\Delta\Theta]_C \tag{27}$$

[1] Călugăreanu's paper is in French; he writes "...la torsion passe nécessairement par un point de discontinuité ..." (Călugăreanu, 1961, p. 616).

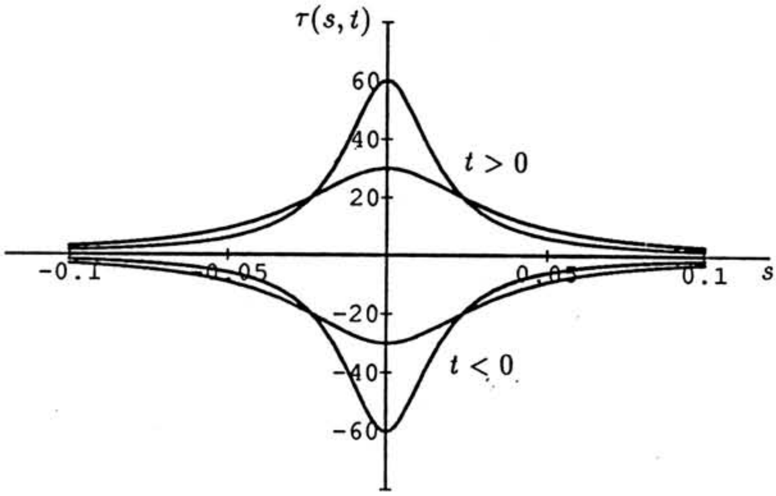

Fig. 8. The torsion function $\tau(s,t)$ as given by (25) for various values of t.

and we write it in this way to emphasise that it is none other than the helicity invariant divided by κ^2. When $\mathcal{C}$ contains an inflexion point, $\mathcal{T}(\mathcal{C})$ and $\frac{1}{2\pi}[\Delta\Theta]_{\mathcal{C}}$ are both indeterminate. In deformation through a configuration containing an inflexion point, both $\mathcal{T}(\mathcal{C})$ and $\frac{1}{2\pi}[\Delta\Theta]_{\mathcal{C}}$ are discontinuous but by equal and opposite amount so that $\mathcal{T}(\mathcal{C}) + \frac{1}{2\pi}[\Delta\Theta]_{\mathcal{C}}$ is continuous; we may then adopt the common limit (as $t \to t_c$ from above or below) to give meaning to $\mathcal{T}(\mathcal{C}) + \frac{1}{2\pi}[\Delta\Theta]_{\mathcal{C}}$ at the moment of discontinuity.

Of course all these subtleties are avoided if we simply adopt the helicity $\mathcal{H} = n\kappa^2$ of the (framed) vortex tube as the fundamental invariant which is insensitive to the presence or absence of inflexion points!

7. Summary and discussion

We have shown that, if the vortex lines in a knotted vortex filament are twisted in such a way that each vortex line is a closed curve which closes after one passage around the tube, and each pair of vortex lines in the tube has linking number n, then the helicity of the vorticity field is given by

$$\mathcal{H} = \int_{\mathcal{V}} \mathbf{u} \cdot \boldsymbol{\omega} \, d\mathcal{V} = n\kappa^2 \tag{28}$$

where κ is the circulation associated with the tube. The integer n is an invariant under frozen field distorsion of the tube, and is identified with the

Călugăreanu (1961) invariant:

$$n = W(\mathcal{C}) + \mathcal{T}(\mathcal{C}) + \frac{1}{2\pi}[\Delta\Theta]_{\mathcal{C}} \tag{29}$$

where the writhing number $W(\mathcal{C})$ and the total torsion $\mathcal{T}(\mathcal{C})$ are defined by eqs. (11) and (12). The angle $[\Delta\Theta]_{\mathcal{C}}$ represents the total angle of rotation of a neighbouring curve $\mathcal{C}'$ (which we here take to be a vortex line) relative to the Frenet pair $(\mathbf{n}, \mathbf{b})$ in one passage around the tube; $\frac{1}{2\pi}[\Delta\Theta]_{\mathcal{C}}$ is an integer and so therefore is $W(\mathcal{C}) + \mathcal{T}(\mathcal{C})$.

If $\mathcal{C}$ is deformed continuously, then it may pass through configurations containing one or more inflexion points. When $\mathcal{C}$ contains an inflexion point (or an odd number of inflexion points), the torsion is singular. This behaviour is analysed in §6 and it is shown that (generically) $\mathcal{T}(\mathcal{C})$ jumps discontinuously through ± 1 as $\mathcal{C}$ passes through the inflexional configuration. By virtue of the invariance of $\mathcal{H}$, there is then a compensating jump of $\mp 2\pi$ in $[\Delta\Theta]_{\mathcal{C}}$. This behaviour is associated with the classical Reidemeister move of type I.

Călugăreanu's proof of the result (29) is long and complicated. It should be possible to derive the result directly from consideration of the limiting form of the helicity integral. We hope to present such a derivation in a future paper.

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