Topology bounds energy of knots and links

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In this paper, we determine two quantities, of geometric and topological character, that were left undetermined in two previous results obtained by Arnold (Arnold 1974 In *Proc. Summer School in Diff. Eqs. at Dilizhan*, pp. 229–256.) and Moffatt (Moffatt 1990 *Nature* **347**, 367–369) on lower bounds for the magnetic energy of knots and links in ideal fluids. For dissipative systems, a lower bound on magnetic helicity in terms of the average crossing number and a new relationship between rates of change of these two quantities are also determined.

Keywords: knots; linking number; crossing number; magnetic energy; helicity

1. Introduction

Relationships between energy and topology of physical systems have captured the imagination of scientists since the time of Lord Kelvin's vortex atom theory (Lord Kelvin 1867). Nowadays, from string theory to DNA biology, the search for establishing such relations for knotted and linked structures, in particular, has received unprecedented attention (e.g. Stasiak *et al.* 1998; Calvo *et al.* 2005). In recent years, for instance, rigorous results based on magnetic energy considerations and knot theory (Arnold 1974; Freedman 1988; Moffatt 1990; Freedman & He 1991; Berger 1993) establish, in the ideal case, lower bounds for the magnetic energy of knots and links in terms of their topological complexity. In these studies, however, two quantities of geometric and topological character have been left undetermined, and the scope of this article is to determine their exact value, providing explicit formulae for both of them. Moreover, in the case of a change of topology due to dissipation, we determine a lower bound on the helicity (Moffatt 1969) of the magnetic field, which measures the linking of the field lines, in terms of the average crossing number, which provides an algebraic measure of the morphological complexity of the system, and the corresponding rates of change of these quantities are found.

2. Magnetic knots and links in an ideal fluid

To investigate the relationship between energy and topology of physical systems, we focus our attention on the magnetic energy of knots and links. This approach offers several advantages and serves as a good prototype for understanding fundamental

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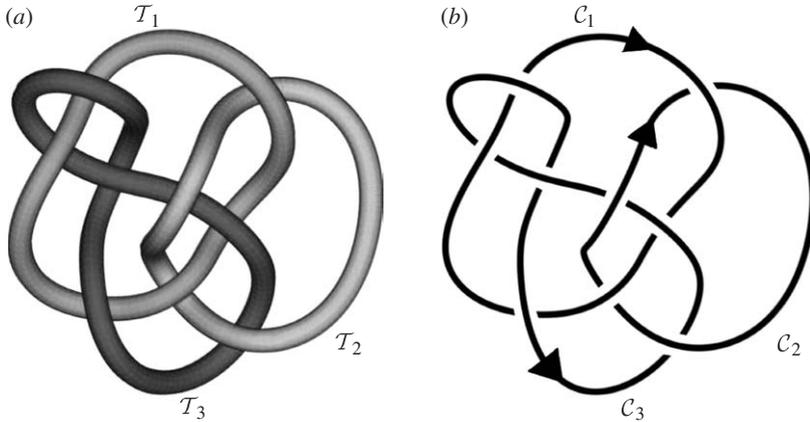[†] Present address: Institute for Scientific Interchange, Villa Gualino, 10133 Torino, Italy.

Figure 1. The three-component link 9^3_{10} with nine minimum number of crossings is represented by (a) a disjoint union of three solid tori and (b) the corresponding oriented centre loops.

issues, relevant also to other disciplines (see, for example, Arnold & Khesin 1998). For this purpose, consider an incompressible and perfectly conducting fluid in $\mathbb{R}^3$. Let $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ be the fluid velocity, smooth function of the position vector $\boldsymbol{x}$ and time t, such that $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$ in $\mathbb{R}^3$ and $\boldsymbol{u} = 0$ at infinity. Consider the class of magnetic fields $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{x}, t)$ that are solenoidal, frozen and of finite energy in $\mathbb{R}^3$, i.e.

$$\boldsymbol{B} \in \{\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \quad \partial_t \boldsymbol{B} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}), \quad L_2\text{-norm}\}. \tag{2.1}$$

In this context, we restrict our attention to magnetic knots and links: by construction (e.g. Moffatt 1990; Ricca 1998), these are tubular embeddings of the magnetic field in nested tori $\mathcal{T}_i$ $(i = 1, \ldots, n)$ centred on smooth oriented loops $\mathcal{C}_i$ that are knotted and linked in the fluid domain (figure 1). Therefore, we identify an n-component magnetic link $\mathcal{L}_n$ with the standard smooth embedding of a disjoint union of n magnetic solid tori in $\mathbb{R}^3$,

$$\sqcup_i \mathcal{T}_i \hookrightarrow \mathcal{L}_n := \mathrm{supp}\,(\boldsymbol{B}). \tag{2.2}$$

The link $\mathcal{L}_n$ is trivial if the loops $\mathcal{C}_i$ bound n smoothly and disjointly embedded discs. Otherwise, the link is *essential*. Let $V = V(\mathcal{L}_n)$ be the total volume of the magnetic link.

We take $\boldsymbol{B} \cdot \boldsymbol{\nu} = 0$ on each tubular boundary $\partial \mathcal{T}_i$ of unit normal $\boldsymbol{\nu}$; the flux Φ_i of the magnetic field through each cross-sectional area S_i of $\mathcal{T}_i$ is given by

$$\Phi_i = \int_{S_i} \boldsymbol{B} \cdot \boldsymbol{\nu} \, \mathrm{d}^2 \boldsymbol{x}. \tag{2.3}$$

Consider the evolution of $\mathcal{L}_n$ under the action of the group of volume- and flux-preserving diffeomorphisms $\varphi : \mathcal{L}_n \to \mathcal{L}_{n,\varphi}$. Two fundamental physical quantities of the system are the magnetic energy and the magnetic helicity, defined respectively by

$$M(t) := \int_{V(\mathcal{L}_n)} \|\boldsymbol{B}\|^2 \, \mathrm{d}^3 \boldsymbol{x}, \quad H(t) := \int_{V(\mathcal{L}_n)} \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}^3 \boldsymbol{x}, \tag{2.4}$$
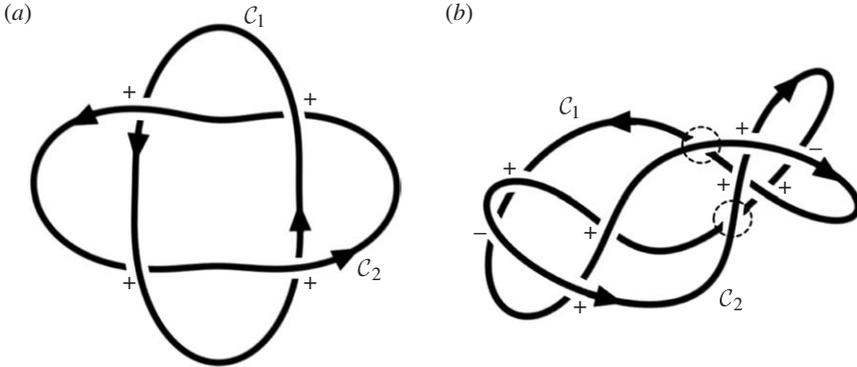
Figure 2. The two-component oriented link 4_1^2 has four minimum number of crossings and the Gauss linking number $Lk_{12}=2$. This, being a topological invariant, does not depend on specific projections or geometric representations: the same link type is shown (*a*) in its minimal projection, with the four crossings denoted by the plus sign, and (*b*) with redundant crossings. Note that the algebraic sum of signed crossings (omitting self-crossings) remains unchanged: the two crossings in (*b*), denoted by dashed circles, do not contribute to the linking number calculation of equation (2.6) because in each case the crossing strands belong to the same link component.

where $\boldsymbol{A}$ is the vector potential associated with $\boldsymbol{B}=\nabla\times\boldsymbol{A}$. We take $\nabla\cdot\boldsymbol{A}=0$ in $\mathbb{R}^3$.

For frozen fields, helicity is a conserved quantity (Woltjer 1958), thus $H(t)=H=$ const. It is known that helicity admits a topological interpretation in terms of linking numbers (Moffatt 1969; Berger & Field 1984; Moffatt & Ricca 1992).

Theorem 2.1. *Let* $\mathcal{L}_n$ *be an essential magnetic link in an ideal fluid. Then,*

$$H = \sum_i Lk_i \Phi_i^2 + 2\sum_{i\neq j} Lk_{ij}\Phi_i\Phi_j, \qquad (2.5)$$

where Lk_i *denotes the Călugăreanu–White linking number of* $\mathcal{C}_i$ *with respect to the framing induced by the embedding of* $\boldsymbol{B}$ *in* $\mathcal{T}_i$, *and* Lk_{ij} *denotes the Gauss linking number of* $\mathcal{C}_i$ *with* $\mathcal{C}_j$.

The Gauss linking number Lk_{ij} is a topological invariant of link types and admits an interpretation in terms of signed crossings: by assigning $\epsilon_r=\pm 1$ to each apparent crossing (following a standard sign convention; figure 2), it can be expressed as the sum over the r signed crossings,

$$Lk_{ij} \equiv \int_{i,j} \mathrm{d}\omega_{i,j} = \frac{1}{2}\sum_{r\in\mathcal{C}_i\sqcap\mathcal{C}_j}\epsilon_r \quad (i\neq j), \qquad (2.6)$$

where $\mathrm{d}\omega_{i,j}$ is the classical Gauss integrand form associated with the two curves $\mathcal{C}_i$ and $\mathcal{C}_j$ and '$\sqcap$' denotes the disjoint union on the apparent intersections of curve strands, omitting self-crossings. The Călugăreanu–White linking number Lk_i is a topological invariant of each link component and admits a geometric decomposition in terms of the writhing number Wr_i and the twist number Tw_i, according to the well-known formula (Călugăreanu 1961; White 1969)

$$Lk_i = Wr_i + Tw_i. \qquad (2.7)$$

The writing number measures the average distortion of $\mathcal{C}_i$ in space, while the total twist measures the total winding of the field lines within $\mathcal{T}_i$.

For simplicity, we assume that all link components have equal flux Φ. The following results hold true.

Theorem 2.2. *Let* $\mathcal{L}_n$ *be an essential magnetic link in an ideal fluid. Then,*

$$\exists^+\text{l.b.} : \lim_{t\to\infty} M(t) = M_{\text{inf}}; \qquad (2.8a)$$

$$M_{\text{min}} = m\frac{\Phi^2}{V^{1/3}}, \qquad (2.8b)$$

where m *is a topological invariant of* $\mathcal{L}_n$.

Equation (2.8a) is due to Freedman (1988) and equation (2.8b) is due to Moffatt (1990). Here M_{inf} denotes the positive infimum value of the magnetic energy m under the limiting (pointwise) process of all possible relaxations of $M(t)$. M_{min}, generally greater than M_{inf}, is the constrained minimum energy attained by topological relaxation of the magnetic field (Moffatt 1985), and m is an isotopy invariant associated with the groundstate energy of the magnetic link. Attempts to determine M_{min}, and therefore m, have been carried out (e.g. by Chui & Moffatt 1995) by applying techniques of constrained magnetic relaxation, but analytical results on the more elusive M_{inf} are still lacking. On the other hand, it is known that the magnetic energy can be bounded from below by field complexity. A useful algebraic measure of complexity is given by the average crossing number $\overline{C}$: quite simply, this is defined as the sum of all unsigned crossings present in the system, averaged over all projections. For two loops $\mathcal{C}_i$ and $\mathcal{C}_j$, we have

$$\overline{C}_{ij} := \int_{i,j} |d\omega_{i,j}| = \left\langle \sum_{r\in\mathcal{C}_i\#\mathcal{C}_j} |\epsilon_r| \right\rangle, \qquad (2.9)$$

where '#' denotes now the disjoint union on all apparent intersections of curve strands, including self-crossings. In the two cases of figure 2, for example, if the loops were lying almost flat on a plane, allowing indentations for over/underpassages of the loop strands, we would have $\overline{C}_{12} = 4$ and $\overline{C}_{12} = 10$ for figure 2a and figure 2b, respectively. Lower bounds on energy are provided by the following results.

Theorem 2.3. *Let* $\mathcal{L}_n$ *be an essential magnetic link in an ideal fluid. Then,*

$$M(t) \geq q|H|, \qquad (2.10a)$$

$$M(t) \geq \left(\frac{16}{\pi}\right)^{1/3} \frac{\Phi^2\,\overline{C}}{V^{1/3}}, \qquad (2.10b)$$

where q *depends on the geometry of* $\text{supp}(\boldsymbol{B})$ *and* $\overline{C}$ *denotes the average crossing number of* $\mathcal{L}_n$.

Here inequality (2.10a) is due to Arnold (1974) and inequality (2.10b) is due to Freedman & He (1991). In theorems 2.2 and 2.3, both m and q are left undetermined. We are now ready to prove the following results.

Theorem 2.4. *Let $\mathcal{L}_n$ be a zero-framed, essential magnetic link, embedded in an incompressible perfect fluid. Then, we have*

$$q = \left(\frac{16}{\pi}\right)^{1/3} \frac{1}{V^{1/3}}, \tag{2.11a}$$

$$m = \left(\frac{16}{\pi}\right)^{1/3} c_{\min}, \tag{2.11b}$$

where $c_{\min}$ is the topological crossing number of $\mathcal{L}_n$.

Proof. The assumption of zero framing implies $Lk_i = 0$, for $i = 1, \ldots, n$. This simplifies equation (2.5) of theorem 2.1 to

$$H = \Phi^2 \left(\sum_i Lk_i + 2\sum_{i \neq j} Lk_{ij}\right) = 2\Phi^2 \sum_{\{i,j\}} Lk_{ij}. \tag{2.12}$$

From the definition of the average crossing number and linking number (equations (2.6) and (2.9)), we have

$$\overline{C} = \sum_{\{i,j\}} \left\langle \sum_r |\epsilon_r| \right\rangle \quad \text{and} \quad \sum_{\{i,j\}} |Lk_{ij}| = \frac{1}{2} \sum_{\{i,j\}} \left| \sum_r \epsilon_r \right|, \tag{2.13}$$

where the external sum is extended to all curves in $\mathcal{L}_n$. Hence,

$$\overline{C} \geq \sum_{\{i,j\}} \left| \sum_r \epsilon_r \right| = 2 \sum_{\{i,j\}} |Lk_{ij}| \geq 2 \left| \sum_{\{i,j\}} Lk_{ij} \right|, \tag{2.14}$$

and therefore

$$|H| = 2\Phi^2 \left| \sum_{\{i,j\}} Lk_{ij} \right| \leq \Phi^2 \overline{C} \quad \text{or} \quad \overline{C} \geq \frac{|H|}{\Phi^2}. \tag{2.15}$$

Now, from equation (2.10*b*), we have

$$M(t) \geq \left(\frac{16}{\pi}\right)^{1/3} \frac{\Phi^2 \overline{C}}{V^{1/3}} \geq \left(\frac{16}{\pi}\right)^{1/3} \frac{\Phi^2 c_{\min}}{V^{1/3}}, \tag{2.16}$$

by the definition of $c_{\min}$ as the minimum number of crossings. In addition, by equation (2.15), we have

$$M(t) \geq \left(\frac{16}{\pi}\right)^{1/3} \frac{\Phi^2 \overline{C}}{V^{1/3}} \geq \left(\frac{16}{\pi}\right)^{1/3} \frac{\Phi^2}{V^{1/3}} \frac{|H|}{\Phi^2} = \left(\frac{16}{\pi}\right)^{1/3} \frac{|H|}{V^{1/3}}. \tag{2.17}$$

By comparing inequality (2.17) with inequality (2.10*a*), equation (2.11*a*) follows. Finally, by considering inequality (2.16) at $M_{\min}$, we have also (2.11*b*). ∎

As expected, equation (2.11*a*) shows that indeed q depends solely on the geometry of the link through the cubic root of its volume, while equation (2.11*b*) shows that m is actually related to the topological crossing number of the link type. By direct inspection of link tabulation, it is immediately evident that there are countably many topologically distinct links with equal number n of (zero-framed) components and same $c_{\min}$ (figure 3). This suggests that framing of each link component becomes a necessary and crucial specification when we want to identify uniquely the groundstate energy of knot/link types with topology.
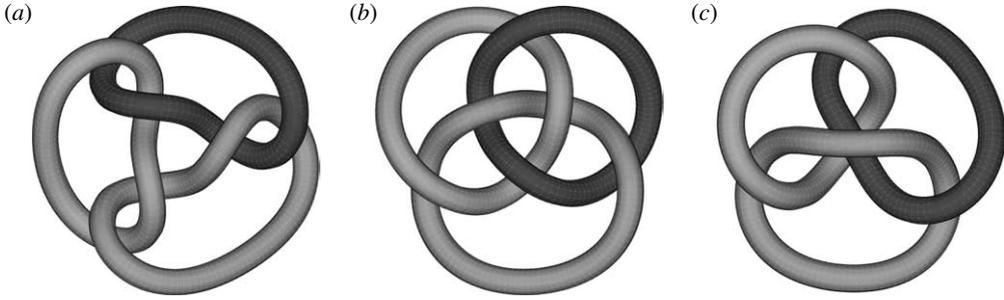
Figure 3. Three distinct link types with $n=3$ and $c_{\min}=6$ ((a) 6_1^3, (b) 6_2^3 and (c) 6_3^3). By assuming zero framing in all link components, same volume V and flux Φ, then by equation (2.11b) the three links must have the same groundstate energy $M_{\min}$. Thus, different prescribed framing is necessary if we want to identify uniquely each knot/link type with its specific groundstate energy.

3. Change of topology in the presence of dissipation

Suppose now that the fluid is no longer perfectly conducting, but resistive. The topology of $\mathcal{L}_n$ may now change due to the effects of dissipation, which make reconnections of the magnetic field lines possible. Under these conditions, magnetic helicity may also change, hence $H = H(t)$. The change in magnetic helicity can be measured in terms of change in algebraic complexity of the magnetic link. For this, let us define $\Omega := \sum_{\{i,j\}} d\omega_{i,j}$; we have the following results.

Theorem 3.1. *Let $\mathcal{L}_n$ be a zero-framed, essential magnetic link, embedded in a resistive, incompressible fluid. Then, we have*

$$|H(t)| \le 2\Phi^2 \overline{C}(t); \tag{3.1a}$$

$$\frac{d|H(t)|}{dt} \le \text{sign}(H)\text{sign}(\Omega)2\Phi^2 \frac{d\overline{C}(t)}{dt}. \tag{3.1b}$$

Proof. Since

$$|H| = \left| 2\Phi^2 \sum_{\{i,j\}} \int_{i,j} d\omega_{i,j} \right| \quad \text{and} \quad \overline{C} = \sum_{\{i,j\}} \int_{i,j} |d\omega_{i,j}|, \tag{3.2}$$

we have

$$|H(t)| = 2\Phi^2 \left| \sum_{\{i,j\}} \int_{i,j} d\omega_{i,j} \right| \frac{\overline{C}(t)}{\sum_{\{i,j\}} \int_{i,j} |d\omega_{i,j}|}$$

$$\le 2\Phi^2 \left| \sum_{\{i,j\}} \int_{i,j} d\omega_{i,j} \right| \frac{\overline{C}(t)}{\left| \sum_{\{i,j\}} \int_{i,j} d\omega_{i,j} \right|} = 2\Phi^2 \overline{C}(t), \tag{3.3}$$

thus (3.1a) is proved. For (3.1b), consider first the elementary change in magnetic helicity, given by

$$d|H(t)| = d[\text{sign}(H)H(t)] = \text{sign}(H)dH(t) = \text{sign}(H)2\Phi^2 \sum_{\{i,j\}} d\omega_{i,j}, \tag{3.4}$$

and the corresponding elementary change in average crossing number, given by

$$\mathrm{d}\,\overline{C}(t) = \sum_{\{i,j\}} |\mathrm{d}\omega_{i,j}|. \tag{3.5}$$

Then, we have

$$\mathrm{d}\,|H(t)| = \mathrm{sign}\,(H)2\varPhi^2 \frac{\sum_{\{i,j\}} \mathrm{d}\omega_{i,j}}{\sum_{\{i,j\}} |\mathrm{d}\omega_{i,j}|} \mathrm{d}\,\overline{C}(t) \le \mathrm{sign}\,(H)2\varPhi^2 \frac{\sum_{\{i,j\}} \mathrm{d}\omega_{i,j}}{|\sum_{\{i,j\}} \mathrm{d}\omega_{i,j}|} \mathrm{d}\,\overline{C}(t)$$

$$= \mathrm{sign}(H)\mathrm{sign}(\varOmega)2\varPhi^2 \,\mathrm{d}\,\overline{C}(t). \tag{3.6}$$

In the limit, by taking the time derivative of (3.6), we have (3.1*b*). Theorem 3.1 is thus proved. ∎

A careful check on direct numerical tests performed on tangle complexity (Barenghi *et al.* 2001) confirms these results: from the data analysis of the tangle mature growth stage (see fig. 5 of that paper; $\varPhi = 1$ in appropriate units), we have $(2\overline{C}(t) - |H(t)|)/|H(t)| \approx 19.3\%$ at $t = 0.09$ (cf. inequality (3.1*a*)), and $\mathrm{sign}(H)\mathrm{sign}(\varOmega) = +$, $(2\Delta\overline{C}(t) - \Delta|H(t)|)/\Delta|H(t)| \approx 27.6\%$ for $t \in [0.08, 0.09]$, both values well above the equalities of theorem 3.1. Since these results are independent of specific viscous or resistive time scales, physical time ought to be interpreted in terms of the reconnection time scale involved in the change of topology.

The results of theorems 2.4 and 3.1 find applications in developing new measures of structural complexity (Ricca 2005, in press) and provide further ground to establish a theoretical foundation for a classification of physical knots and links based on a one-to-one correspondence between energy and topology.

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