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A new stretch-twist-fold model for fast dynamo

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Preliminary results on a new Stretch-Twist-Fold (STF) kinematic model for fast dynamo are presented. The evolution is prescribed by equations that govern the simultaneous stretching, writhing and coiling of a magnetic flux-tube by diffeomorphism of the initial circular configuration. Simple estimates based on minimized magnetic energy show that exponential growth of the magnetic field is indeed possible.

1 Kinematic modeling of the Stretch-Twist-Fold (STF) cycle

Spontaneous generation, growth and sustainment of magnetic field by dynamo action is an open and fundamental problem in the study of geo and planetary magnetic fields. An appealing kinematic model for fast dynamo was proposed by Vainshtein & Zeldovich in 1974: in this model growth of magnetic field is realized by an iterated sequence of idealized stretch-twist-fold (STF) process and diffusive relaxation. In the first, ideal stage the STF cycle is performed by volume- and flux-preserving diffeomorphism of a magnetic flux-tube. Under conservation of volume, stretching induces a decrease of the average tube cross-section, while flux conservation yields simultaneous magnetic field growth; during this process the flux-tube is conveniently twisted and folded on itself to maintain the original spatial scaling. During the diffusive stage, then, resistive effects induce the merging of the magnetic tube elements to form a new, stronger flux-tube.

Several ideas behind the STF cycle have been extensively investigated (see the monograph by Childress & Gilbert, 1995), but so far an accurate modeling of the original Vainshtein & Zeldovich's STF cycle has been rather elusive. A notable exception is the study proposed by Moffatt & Proctor's (1985), based on the realization of an appropriate flow map from a decomposition of the velocity field in separate, distinct actions. What we provide here is the first accurate reproduction of the full STF cycle by continuous diffeomorphism of an initial circular configuration, hence mimicking very closely the original idea of Vainshtein and Zeldovich. To this end we make use of a new set of kinematic equations (see Maggioni and Ricca, 2006, and our sister paper in these Proceedings), that govern the evolution of the flux-tube axis $\mathbf{X} = \mathbf{X}(\xi, t)$. This is given by

$$\mathbf{X} = \mathbf{X}(\xi, t) \ : \quad \begin{cases} x = F_1(t)\cos(m\xi) - G_1(t)\cos(n\xi) \\ y = F_2(t)\sin(m\xi) - G_2(t)\sin(n\xi) \\ z = G_3(t)\sin(\xi)\ , \end{cases} \tag{1}$$

where we take $F_1(t) = F_2(t) = \mathrm{Sech}(t)$, $G_1(t) = G_2(t) = G_3(t) = t^{1/100}$ and $\{n, m\}$ ($n > m > 0$; n, m integers) control the number of initial coverings and coils. Since time is here purely kinematical, the present choice of time-dependence is entirely arbitrary.

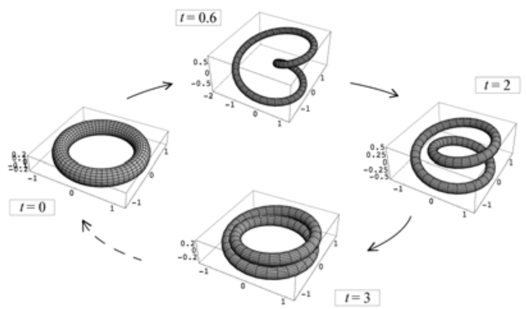

Fig. 1 STF prototype cycle by model equations (1),with $m = 1$ and $n = 2$: an initially circular tube ($t = 0$) gets stretched, while writhing and coiling-up in space, before double-covering the circle ($t = 3$) towards the initial configuration. In the diffusive stage (indicated by a dashed arrow) resistive effects induce the merging of the tube strands, to generate a new, single flux-tube of increased intensity.

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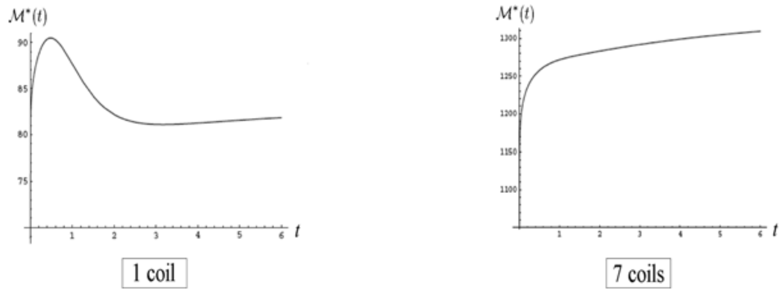

Fig. 2 Growth of minimum energy against time. In the case of 1 coil, $\mathcal{M}^*$ shows an inflexional behavior at about $t = 1.2$, by dropping from a maximum to a minimum, to grow again after about $t = 3.1$. For a larger number of coils, for example 7, $\mathcal{M}^*$ is monotonically increasing in the same time range.

Figure 1 shows one STF cycle produced by the model equations (1), with $m = 1$ and $n = 2$. The full process is completed by a diffusive stage (indicated by dashed arrow, but not described by model equations 1), in which the double-stranded configuration is merged into a newly intensified flux-tube. In the cycle shown only one coil is produced, but in a similar way any $n > 2$ would give $n - 1$ coils simultaneously generated (see Maggioni & Ricca, 2006 for a detailed description of model equations). The velocity field $\mathbf{u}(\xi, t) = D\mathbf{X}(\xi, t)/Dt$, then, can be straightforwardly computed by the components of (1) and adjusted in relation to the real dynamics (work in progress).

2 Growth of minimum energy by model equations

Detailed analysis on the magnetic energy has been carried out for a thin, standard flux-tube, with prescribed magnetic field, given by a toroidal and a poloidal component, according to the approach adopted by Ricca (2005). Assuming that i) the cross-section of the flux-tube remains circular and uniform along the tube axis, ii) the flux-function is periodic and independent of arc-length, iii) the twist h of the magnetic field lines — given by the ratio of poloidal to toroidal flux — remains uniform along the tube axis, then minimization of magnetic energy gives

$$\mathcal{M}^* = \frac{1}{\epsilon^2}\left(\frac{\Phi^2}{2\pi L} + \epsilon^2 \frac{\pi h^2 \Phi^2}{L}\right) , \tag{2}$$

where $\epsilon \ll 1$ denotes tube aspect ratio, L total length of tube axis, Φ total magnetic flux. Equation (2) is exact at all orders of ϵ, and improves previous result by Chui & Moffatt (1995). Consider now the STF cycle governed by eqs. (1): under stretching $L = L(t)$ increases; by taking tube volume $V = 1$ (constant) and flux $\Phi = 1$ (constant), eq. (2) reduces to

$$\mathcal{M}^*(t) = \frac{L^2(t)}{2} + \frac{\pi h^2}{L(t)} , \tag{3}$$

where now $L(t)$ is given by eqs. (1). This gives the minimum energy of the flux-tube at each time step of the STF cycle given by eqs. (1). Figure 2 shows that the minimized energy can grow super-exponentially even in the case of a single coil formation. Mean exponential growth of energy can then attain realistic values for fast dynamo by convenient adjustment (in the model equations) of stretching and coiling formation.

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