

INTERPRETATION OF INVARIANTS OF THE BETCHOV-DA RIOS EQUATIONS AND OF THE EULER EQUATIONS

H. Keith Moffatt and Renzo L. Ricca

Department of Applied Mathematics and Theoretical Physics
Silver Street, Cambridge CB3 9EW, U.K.
and
Trinity College
Cambridge CB2 1TQ, U.K.

INTRODUCTION

Interest in the study of invariant quantities is generally motivated by the need to interpret and to understand their meaning and their fundamental role in the theory. The invariants we shall consider in this paper emerge in two contexts. In the context of the localized induction approximation (LIA) for the motion of an inextensible vortex filament in a perfect fluid flow, we shall deal with certain conserved quantities that emerge from the Betchov-Da Rios equations; the polynomial invariants for the related nonlinear Schrödinger equation (NLSE) are calculated employing the recurrence formula of Zakharov and Shabat (1974); these quantities are constants of the motion for the vortex as long as self-intersection does not occur and long-distance effects are neglected. Some of them are then interpreted in terms of kinetic energy, momentum, angular momentum, "pseudo-helicity," and some inequalities between physical and global geometrical properties are stated. In the context of the Euler equations, given that the topology of the vortex structure is conserved and the helicity invariant is the natural measure of the topological complexity of the field (Moffatt, 1969; Arnol'd, 1974), we comment further on this role of helicity; moreover, by associating an "energy spectrum" with any knot or link present in the fluid, the lowest ground-state enstrophy is interpreted as an even more powerful invariant for evaluating the knot or link complexity of the field structure.

INVARIANTS OF THE BETCHOV-DA RIOS EQUATIONS

We consider a vortex filament \mathcal{F} in a perfect fluid flow, with the vorticity vector ω everywhere parallel to the tangent \mathbf{t} on the curve. In the context of the local induction approximation (LIA) the motion of the vortex is governed by the simple equation

$$\mathbf{V} = c(s, t)\mathbf{b} \quad (1)$$

where \mathbf{V} is the Eulerian velocity of \mathcal{F} , $c(s, t)$ is the local curvature, a function of arclength s and time t , and \mathbf{b} the binormal (Da Rios, 1906; Arms and Hama, 1965). In the LIA context,

the mean diameter of the vortex core-size is assumed negligible compared with the local radius of curvature and self-interactions of different parts of the filament are believed not to occur. The equation (1) led Da Rios and Betchov (1965), independently, to derive the equations which govern the time derivative of the curvature c and the torsion ϑ of \mathcal{F}

$$\left. \begin{aligned} \dot{c} &= -(c\vartheta)' - c'\vartheta \\ \dot{\vartheta} &= \left(\frac{c'' - c\vartheta^2}{c} \right)' + c'c \end{aligned} \right\} \quad (2)$$

where dots and primes stand for partial derivatives with respect to t and s , respectively. Hasimoto (1972) by the remarkable transformation

$$\psi(s, t) = c(s, t) \exp \left[i \int \vartheta(s, t) ds \right] \quad \psi(s, t) \in \mathbb{C} \quad (3)$$

reduced the set (2) to the nonlinear Schrödinger equation (NLSE)

$$\frac{1}{i} \frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial s^2} + \frac{1}{2} |\psi|^2 \psi \quad (4)$$

which in one dimension is completely integrable (Zakharov and Shabat, 1972). This means that we have an infinite set of quantities which are constants of the motion. Among these, there is a countable family of so-called 'polynomial conservation laws'; they have the form of an integral with respect to s of a polynomial expression in terms of the function $\psi(s, t)$ and its derivatives with respect to s . Using the Hasimoto transformation (3), we can employ the recurrence formula given by Zakharov and Shabat to calculate this family of invariants in terms of c and ϑ . The formula is

$$f_{n+1} = q \frac{\partial}{\partial s} \left(\frac{1}{q} f_n \right) + \sum_{j+k=n} f_j f_k \quad ; \quad f_n = f_n(s, t) \in \mathbb{C} \quad n = 1, 2, \dots \quad (5)$$

where

$$q = \frac{1}{2} i \psi \quad f_1 = \frac{1}{4} |\psi|^2 = \frac{1}{4} c^2$$

and the associated invariants are then

$$(2i)^n \mathcal{I}_n = \oint f_n(s, t) ds = \text{constant} \quad n = 1, 2, \dots \quad (6)$$

The first five conserved quantities are

$$2i\mathcal{I}_1 = \frac{1}{4} \oint |\psi|^2 ds = \frac{1}{4} \oint c^2 ds \quad (7a)$$

$$(2i)^2 \mathcal{I}_2 = -\frac{1}{8} \oint (\bar{\psi} \psi' - \psi \bar{\psi}') ds = -\frac{1}{4} i \oint c^2 \vartheta ds \quad (7b)$$

$$(2i)^3 \mathcal{I}_3 = \frac{1}{4} \oint \left(\frac{c^4}{4} - c'^2 - c^2 \vartheta^2 \right) ds \quad (7c)$$

$$(2i)^4 \mathcal{I}_4 = \frac{1}{4} \oint \left[cc''' - i \left(\frac{3}{4} c^4 - c'^2 - c^2 \vartheta^2 + 2cc'' \right) \vartheta \right] ds \quad (7d)$$

$$(2i)^5 \mathcal{I}_5 = \frac{1}{4} \oint \left[c^2 \left(\frac{c^4}{8} - c'^2 \right) - \frac{3}{2} cc'^2 + c''^2 + c^2 \left(\vartheta^4 - \frac{3}{2} c\vartheta + \vartheta'^2 \right) - 2\vartheta^2 (2cc'' + c'^2) \right] ds. \quad (7e)$$

where $\bar{\psi}$ is the complex conjugate of ψ . We recognise the first two integrals (7a, 7b) as invariants discovered by Betchov; note that the total torsion of the filament

$$\mathcal{I}_T = \oint \vartheta ds \quad (8)$$

is also an invariant of equations (2) (Keener, 1990 — note added in proof).

INTERPRETATION OF SOME SCALAR AND VECTOR INVARIANTS OF THE BETCHOV-DA RIOS EQUATIONS

Suppose for simplicity that the vorticity is uniform across the core of the vortex, i.e. $\omega = \omega_0 \mathbf{t}$, and let

$$\Gamma = \omega_0 A_0, \quad d\mathcal{V} = A_0 ds = \frac{\Gamma}{\omega_0} ds$$

where A_0 is the mean cross-sectional area. Apart from numerical coefficients, the integral \mathcal{I}_1 can be related to the Eulerian kinetic energy T associated with the motion of the filament simply applying LIA (eq. 1):

$$T = \frac{1}{2} \int_{\mathcal{F}} \mathbf{V}^2 d\mathcal{V} = \frac{1}{2} \frac{\Gamma}{\omega_0} \oint c^2 ds = \text{constant}. \quad (9)$$

The interpretation of \mathcal{I}_2 is more subtle: it is asymptotically related to the helicity of \mathcal{F} ; in this context we can call it “pseudo-helicity” and write

$$\tilde{\mathcal{H}} = \int_{\mathcal{F}} \mathbf{V} \cdot \text{curl} \mathbf{V} d\mathcal{V} = -2 \frac{\Gamma}{\omega_0} \oint c^2 \vartheta ds = \text{constant} \quad (10)$$

[hint: from the definition, apply LIA and consider the “abnormality” $\mathbf{b} \cdot \text{curl} \mathbf{b}$ of unit binormals to the filament; the surface swept out by the vortex filament minimises the Hamiltonian for

the system (2) (Weatherburn, 1927, p. 197; Marris and Passman, 1969; Rasetti and Regge, 1975; Ricca, in preparation)]. The integrand of \mathcal{I}_3 can be recognised as the Lagrangian density for the "lossless" NLSE, via the Hasimoto transformation (3).

Furthermore, the LIA implies that the total length L and the enstrophy Ω of \mathcal{F} are conserved. By the Schwarz integral inequality we can state the following constraints: the helicity \mathcal{H} has an upper bound, given by

$$\mathcal{H}^2 \leq \Gamma^2 L \oint c^2 ds = \text{constant}. \quad (11)$$

For an unknotted filament, the total curvature is bounded as follows

$$2\pi \leq \oint c ds \leq \left(L \oint c^2 ds \right)^{1/2} = \text{constant} \quad (12)$$

[hint: apply Fenchel theorem (do Carmo, 1976, p. 399)] and for a knotted filament, we have

$$4\pi \leq \oint c ds \leq \left(L \oint c^2 ds \right)^{1/2} = \text{constant} \quad (13)$$

[hint: apply Fary-Milnor theorem (do Carmo, 1976, p. 402)].

It is also possible to prove the conservation of two vector quantities which can be interpreted as momentum invariants; the linear momentum

$$\mathbf{M} = \frac{1}{2} \int_{\mathcal{F}} \mathbf{X} \wedge \boldsymbol{\omega} dV = \frac{1}{2} \Gamma \oint \mathbf{X} \wedge \mathbf{X}' ds \quad (14)$$

and the angular momentum

$$\mathbf{P} = \frac{1}{3} \int_{\mathcal{F}} \mathbf{X} \wedge (\mathbf{X} \wedge \boldsymbol{\omega}) dV = \frac{1}{3} \Gamma \oint \mathbf{X} \wedge (\mathbf{X} \wedge \mathbf{X}') ds \quad (15)$$

are constants of the motion [hint: use LIA and the "expansion theorem" for the vector triple product]. The first of these last two results in particular (the conservation of the right-hand side of (14)) was obtained by Arms and Hama (1965) but the interpretation given here is believed to be new. These results imply that the localized induction approximation respects conservation of both momentum and angular momentum.

TOPOLOGICAL INVARIANTS OF THE EULER EQUATIONS

Since, under evolution governed by the Euler equations, the vorticity field is frozen in the fluid, the complete topology of the vorticity field is conserved. It is interesting to inquire what is the complete set of topological invariants that characterise this topology. One subset of invariants are the helicity invariants (Moreau, 1961; Moffatt, 1969), one such invariant being defined for every closed surface within the fluid on which the normal component of

vorticity is zero. The helicity of linked vortex tubes is directly related to the Gauss linking number, the most basic topological invariant of two linked curves (see figure 1). However, there exist simple linkages (e.g. the Whitehead link of figure 1c) for which the linking number is zero; if an untwisted vortex tube of small cross-section is constructed around each of the curves of figure 1c, then the helicity of the resulting flow is zero, although the topology is obviously non-trivial. The question then arises as to whether there is any other topological invariant of the vorticity field which may provide a measure of the topological complexity of the structure. A possible approach to the problem, which is related to the fundamental problem of the topological classification of knots and links in \mathbb{R}^3 has been described by Moffatt (1990). Here, we simply indicate the essential features of this approach, in the context of the Euler equations.

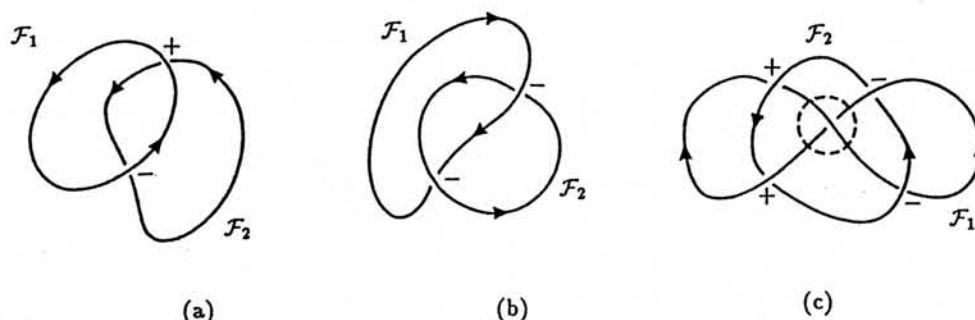


Fig. 1. Examples of (a) unlinked and (b), (c) linked filaments \mathcal{F}_1 and \mathcal{F}_2 . The linking number \mathcal{L}_{12} is respectively: (a) $\mathcal{L}_{12} = 0$; (b) $\mathcal{L}_{12} = -1$; (c) $\mathcal{L}_{12} = 0$ (Whitehead link) (see for example Maxwell, 1873, vol. II, p. 41).

Suppose first that we have an arbitrary closed knotted curve C , and let us construct around C a vortex tube of circulation κ , and small cross-section. We suppose that, within this tube, each vortex line runs essentially parallel to the axis C . There is a certain arbitrariness here as regards the twist of the vorticity field around the axis of the tube: for we can imagine cutting the tube at any section, twisting it about the axis through an arbitrary angle, and reconnecting. As shown in Moffatt (1990), the effect of this is to change the helicity of the associated flow by an amount $h\kappa^2$ where $2\pi h$ is the angle of twist. Whatever the initial helicity may have been, there is always a twist which can be introduced in this way, and which makes the resulting helicity zero. Let us suppose that this condition is satisfied, although it is by no means the only possibility. This assumption pins down the topology of the vorticity field and relates it unambiguously to the topology of the knot C .

Under any volume-preserving diffeomorphism, the topology of the vorticity field is preserved, but the enstrophy of the associated flow is not conserved. It is obvious that there can be no theoretical *upper* bound on the enstrophy, because natural evolution governed by the Euler equations is generally such as to increase enstrophy, and this increase can proceed without limit when the effects of viscosity are ignored; this type of behaviour is well known, if not fully understood, for turbulent flow.

However, it is relevant to ask whether, given the topology of the vorticity field, there is any *lower* bound to the enstrophy, under the action of volume-preserving diffeomorphisms.

If we were to integrate the Euler equations backwards in time, it might be natural in general to expect a decrease of enstrophy, and the question is how far can this decrease proceed, or equivalently what is the minimum level of enstrophy from which flow with given vorticity topology may evolve?

It was shown by simple application of the Schwarz inequality (Moffatt, 1969, section 4) that when the helicity of a flow is non-zero, the enstrophy under Euler evolution (whether forwards or backwards in time) is bounded below, essentially because indefinite contraction of any vortex ring necessarily leads to ultimate extension of any vortex ring with which it is linked. This is a simple and illuminating example of the way that a topological barrier implies the existence of a lower bound. As shown by Freedman (1988) even when the helicity is zero (as for the Whitehead link topology) the enstrophy has a lower bound, under the action of volume-preserving diffeomorphisms, provided only that the topology of the vorticity field is non-trivial. This important result provides a crucial link between formal topological techniques and fluid mechanics.

A technique by which the lower bound may be attained was developed by Moffatt (1985), and is applied in the context of knots and links in Moffatt (1990). The technique is more easily comprehended in terms of the magnetic field in a perfectly conducting fluid rather than in terms of vorticity, but the end result is the same. In terms of the magnetic field, the volume-preserving diffeomorphism is seen as the effect of an incompressible flow which is driven by the Lorentz force associated with the magnetic field. This construction is such as to dissipate magnetic energy (the analogue of enstrophy) and to drive the magnetic field towards a magnetostatic equilibrium state, and in this state, the magnetic energy is minimal with respect to arbitrary volume-preserving frozen-field distortions. The flow that takes the initial magnetic field to this final equilibrium state defines also a limit volume-preserving mapping $x \rightarrow X$ which relates the final position of fluid particles to their initial position.

If we now think in terms of vorticity rather than magnetic field, the same mapping converts the vorticity field (via the usual Cauchy transformation $\omega_i(X) = \omega_{j0}(x) \partial X_i / \partial x_j$) to a state of minimum enstrophy.

For the case in which the vorticity is confined to a single closed knotted vortex tube, as described above, the minimum enstrophy must, on dimensional grounds, have the form

$$\Omega = m\kappa^2 \mathcal{V}^{-\frac{1}{3}} \quad (16)$$

where \mathcal{V} is the volume of the vortex tube and κ the circulation (both conserved under volume-preserving frozen-field distortions). The coefficient m depends only on the topology of the tube, i.e. on the topology of the knot C around which it is constructed. This dimensionless positive real number is evidently a topological invariant of the knot.

The situation becomes a little more complicated when we consider linked vortex tubes. If each tube has the same circulation κ and the same volume \mathcal{V} , then the minimum enstrophy state is characterised by just the two dimensionless parameters κ and \mathcal{V} , and by the topology of the linkage, and so the minimum enstrophy again has the form $m\kappa^2 \mathcal{V}^{-\frac{1}{3}}$ where the real number m is a topological invariant characteristic of the linkage. If however the circulations are different, then the conclusion is not so simple. For two tubes of circulation κ_1, κ_2 exhibiting an arbitrary linkage (e.g. the Whitehead linkage) each tube having volume \mathcal{V} , the enstrophy, on dimensional grounds, has the form

$$\Omega = \Sigma m_{ij} \kappa_i \kappa_j \mathcal{V}^{-\frac{1}{3}} \quad (17)$$

where m_{ij} is a real symmetric positive-definite matrix. Under frozen-field deformation, this matrix varies, and, for given κ_1, κ_2 , there is a unique minimising configuration. However, m_{ij}

in the minimum enstrophy state is a function of the ratio κ_1/κ_2 . This may be seen explicitly for the Whitehead link, for which m_{11} is non-zero if κ_2 is non-zero. However, if κ_2 is zero (so that the second vortex tube vanishes), then m_{11} in the minimum enstrophy state is zero, because the topology is trivial.

Many questions on the frontier between fluid mechanics and topology are raised by considerations of this kind, and these present challenging problems for the future.

CONCLUDING REMARKS

Some comments need to be made in conclusion. Either in the LIA context or in the context of the Euler equations, global geometrical considerations play a critical part. In the LIA context the topology is not conserved; but the invariants are constants of the motion for the isolated vortex filament as long as it does not interact with itself and the long-distance effects are negligible; under these circumstances these invariants are consistent with the assumed model. Some of these conserved quantities have been interpreted in terms of kinetic energy, momentum, angular momentum, "pseudo-helicity," and some inequalities between physical and global geometrical properties have been stated. It is remarkable that even under this approximation the momentum and the angular momentum are still conserved.

In the context of the Euler equations the topology is fully conserved. Helicity, being the natural and simplest measure of topological complexity of a convected vector field, plays a prominent role. Here, we can reasonably expect to have a wider family of invariants of such a topological nature. Adopting the method of magnetic relaxation, which conserves vorticity topology, a new type of invariant has been found; it has been interpreted as the minimum enstrophy state for knotted vortices and its nonzero value is due to the topological barrier in the relaxed field.

Finally, we observe that this invariant should be computable, at least for reasonably simple knots. Given the equation of the knot in parametric form $\mathbf{x} = \mathbf{x}(s)$, it should be straightforward to construct a knot field $B_C(\mathbf{x})$ and then to implement the relaxation procedure numerically. In the equilibrium end-state, delicate questions concerning tangential discontinuities (i.e. vortex sheets or vortex gradient sheets) may arise. This means that particular care has to be taken in the asymptotic stage of relaxation.

REFERENCES

- Arms, R. J., and Hama, F. R., 1965, Localized-Induction Concept on a Curved Vortex and Motion of an Elliptic Vortex Ring, *Phys. Fluids*, 8(4):553.
- Arnol'd, V. I., 1974, The asymptotic Hopf invariant and its applications, (in Russian), in: "Proc. Summer School in Differential Equations," Armenian S.S.R. Acad. Sci., Erevan. (English translation: *Sel. Math. Sov.*, 5:327).
- Betchov, R., 1965, On the curvature and torsion of an isolated vortex filament, *J. Fluid Mech.*, 22:471.
- Da Rios, L. S., 1906, On the motion of an unbounded fluid flow with an isolated vortex filament, (in Italian), *Rend. Circ. Mat. Palermo*, 22:117.
- do Carmo, M. P., 1976, "Differential Geometry of Curves and Surfaces," Prentice-Hall, Englewood Cliffs.
- Freedman, M. H., 1988, A note on topology and magnetic energy in incompressible perfectly conducting fluids, *J. Fluid Mech.*, 194:549.
- Hasimoto, H., 1972, A soliton on a vortex filament, *J. Fluid Mech.*, 51:477.
- Keener, J. P., 1990, Knotted vortex filaments in an ideal fluid, *J. Fluid Mech.*, 211:629.

- Marris, A. W., and Passman, S. L., 1969, Vector Fields and Flows on Developable Surfaces, Arch. Rat. Mech. Anal., 32:29.
- Maxwell, J. C., 1873, "A Treatise on Electricity and Magnetism," MacMillan & Co., Oxford.
- Moffatt, H. K., 1969, The degree of knottedness of tangled vortex lines, J. Fluid Mech., 35:117.
- Moffatt, H. K., 1985, Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology. Part 1. Fundamentals, J. Fluid Mech., 159:359.
- Moffatt, H. K., 1990, The energy spectrum of knots and links, Nature, (to appear).
- Moreau, J. J., 1961, Costantes d'un îlot tourbillonnaire en fluid parfait barotrope, C. R. Acad. Sci. Paris, 252:2810.
- Rasetti, M., and Regge, T., 1975, Vortices in He II, Current Algebras And Quantum Knots, Physica, 80(A):217.
- Weatherburn, C. E., 1927, "Differential Geometry of Three Dimensions," C.U.P., Cambridge.
- Zakharov, V. E., and Shabat, A. B., 1972, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, Sov. Phys. JETP, 34(1):62.