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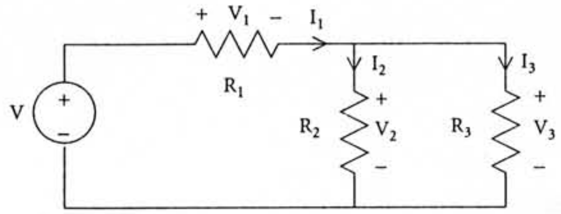

Figure 2. A resistor network.

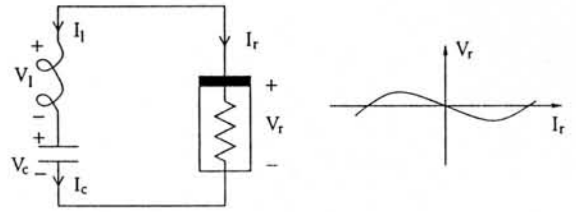

Figure 3. A simple oscillator circuit.

circuit includes inductors and capacitors, the equations become integro-differential, but Kirchhoff's laws work just as well, and they also handle nonlinear components easily. The circuit of Figure 3, for instance, is an example of the van der Pol oscillator. In this case, KCL simply states that the currents through all three elements are identical. KVL implies that $V_l + V_c - V_r = 0$. The constitutive relations for the capacitor and inductor are $I_c = C\,\mathrm{d}V_c/\mathrm{d}t$ and $V_l = L\,\mathrm{d}I_l/\mathrm{d}t$, and the resistor in this circuit is nonlinear: $V_r = -aI_r + bI_r^3$. Thus, KVL can be rewritten as

$$L\frac{\mathrm{d}I_l}{\mathrm{d}t} + \frac{1}{C}\int I_c\mathrm{d}t + aI_r - bI_r^3 = 0.$$

Other Applications

These ideas generalize beyond electronic circuit analysis. Many other systems, ranging from vehicle suspensions to social groups, can be described by networks. Moreover, KVL and KCL are actually instances of a more general set of laws. In the late 1950s and early 1960s, inspired by the realization that the principles underlying KCL and Newton's third law were identical (summation of {forces, currents} at a point is zero, respectively; both are manifestations of the conservation of energy), researchers began combining multi-port methods from a number of engineering fields into a generalized engineering domain with prototypical components (Paynter, 1961). The basis of this generalized physical networks (GPN) paradigm is that the behavior of an ideal two-terminal element—the "component"—can be described by a mathematical relationship between two dependent variables: generalized flow and generalized effort, where flow × effort = power. This pair of variables

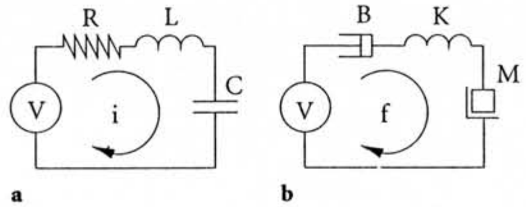

Figure 4. (a) An electrical circuit that is mathematically equivalent to (b) a mechanical circuit.

is different in each domain: (flow, effort) is (current, voltage) in an electrical domain and (force, velocity) in a mechanical domain.

The GPN representation brings out similarities between components and properties in different domains. Electrical resistors ($v = iR$) and mechanical dampers or "dashpots" ($v = fB$) are analogous, as both dissipate energy. Both of the networks in Figure 4, for example, can be modeled by a series inertia-resistor-capacitor GPN. Thus network (a) is an electronic RLC circuit (like the van der Pol example of Figure 3), and network (b) is a mechanical mass-spring-damper system that has identical behavior. Similar analogies exist for generalized inertia, capacitance, flow, and effort source components for mechanical rotational, hydraulic, and thermal domains (Karnopp et al., 1990, Sanford, 1965). These correspondences and generalizations allow applications of KVL and KCL to mechanical structures (buildings, vehicle suspensions, aircraft, etc.), which can be modeled as interconnected networks of masses, springs, and dashpots. This approximation gives analytic insight into the vibrational modes of buildings (important for earthquake protection) and of aircraft (to keep engine frequencies from damaging wings).

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KLEIN–GORDON EQUATION

See **Sine-Gordon equation**

KNOT THEORY

A simple closed curve in three-dimensional space is a knot; more precisely, if M denotes a closed orientable three-manifold, then a smooth embedding of S^1 in M is called a knot in M. A link in M is a finite collection of disjoint knots, where each knot is a component of the link. Knot theory deals with the study and application of mathematical properties of knots and links in pure and applied sciences. As purely mathematical objects, knots are studied for the purpose of classifying three-dimensional surfaces according to the degree of topological complexity, regardless of their specific embedding and geometric properties (Figure 1). In this sense, knot theory is part of topology. In recent years, however, knot theory has embraced applications in dynamical systems, stimulated by the challenging difficulties associated with the study of physical knots (Kauffman, 1995). In this context knots and links are representatives of virtual and numerical objects (given by dynamical flows, phase space trajectories, and visiometric patterns), and are used to model tube-like physical systems, such as vortex filaments, magnetic loops, electric circuits, elastic cords, or even high-energy strings. For physical knots, topological issues and geometric and dynamical aspects are intimately related, influencing each other in a complex fashion. Virtual or numerical knots are studied in relation to the generating algorithms and the probability of forming knots, whereas the study of physical knots addresses questions relating topology and physics, as in the case of the topological quantum field theory (Atiyah, 1990) and topological fluid mechanics (Arnol'd & Khesin, 1998; Ricca, 2001).

Mathematical Aspects

Let us introduce some basic mathematical concepts (see, for example, Adams, 1994). A knot is said to be oriented in M, if it is a smooth embedding of an oriented curve. Two knots K and K' are said to be equivalent if there exists a smooth orientation-preserving automorphism $f : M \to M$ such that $f(K) = K'$; in particular, if the knot K is continuously deformed by f (preserving the curve orientation) to the knot K', then the two knots K and K' are said to be equivalent by ambient isotopy, and the isotopy class of K is represented by its knot type. Since knot theory deals essentially with the properties of knots and links up to isotopy, the knot parametrization, as well as any other geometric information, is irrelevant. A knot diagram of K is a plane projection with crossings marked as under or over; among the infinitely many diagrams representing the same knot K, the minimal diagram is the diagram with a minimum number of crossings. According to the type of crossing, it is customary to assign to each crossing in the knot diagram the value $\varepsilon = +1$ or $\varepsilon = -1$, as shown in Figure 2: by switching one crossing in the knot diagram from positive to negative (or the other way round), we obtain a different knot type, which is identical except for this crossing. By switching all the crossings we obtain the mirror image of the original knot. If the knot is isotopic to its mirror image, then its knot type is said to be achiral, otherwise it is chiral.

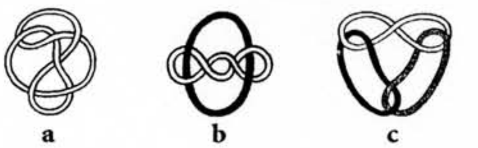

Figure 1. Three examples of knot and link types: (a) the six-crossing knot 6_3; (b) the two-component six-crossing link 6_3^2; (c) the three-component seven-crossing link 7_1^3.

Figure 2. Standard crossing notation and algebraic sign convention for oriented strands: $\varepsilon(K_-) = -1$; $\varepsilon(K_0) = 0$; $\varepsilon(K_+) = +1$.

A knot invariant is a quantity whose value does not change when it is calculated for different isotopic knots. There are many types of invariants of knots and links, but the most common are of numerical or algebraic nature. One of the most important is the genus $g(K)$ of the knot K: recall that closed orientable surfaces are classified by genus, given by the number of handles in a handle-body decomposition. The genus $g(K)$ is defined as the minimum genus over all orientable surfaces S, which span an oriented knot K, where $\partial S = K$. One of the simplest combinatorial invariants of a knot is the minimum number of crossings of a knot K in any projection, called the crossing number $c(K)$. A fundamental invariant of links is the linking number $\mathrm{Lk}(K_1, K_2)$, that measures the topological linking between the knots K_1 and K_2; this invariant, discovered by Carl Friedrich Gauss in 1833, can be easily calculated by the crossing sign convention of Figure 2:

$$\mathrm{Lk}(K_1, K_2) = \frac{1}{2} \sum_{r \in K_1 \sqcap K_2} \varepsilon_r , \qquad (1)$$

where $\varepsilon_r = \pm 1$ and $K_1 \sqcap K_2$ denotes the total number of crossings (not necessarily minimal) between K_1 and K_2. Following the pioneering work of James W. Alexander, who used a Laurent polynomial $\Delta_K(q)$ in q to compute a polynomial invariant for the knot K by using its projection on a plane, many other polynomial invariants have been introduced; most notably the Jones polynomial $V_K(t)$ in $t^{1/2}$, defined by the following set of axioms:

(i) Let K and K' be two oriented knots (or links), which are ambient isotopic. Then

$$V_K(t) = V_{K'}(t). \qquad (2)$$

(ii) If U is the unknotted loop (that is the unknot), then

$$V_U(t) = 1. \quad (3)$$

(iii) If K_+, K_-, and K_0 are three knots (links) with diagrams that differ only as shown in the neighborhood of a single crossing site for K_+ and K_- (see Figure 2), then the polynomial satisfies the following skein relation

$$t^{-1}V_{K_+}(t) - tV_{K_-}(t) = (t^{1/2} - t^{-1/2})V_{K_0}(t). \quad (4)$$

An important property of the Jones polynomial (which is not shared by previous polynomials) is that it can distinguish between a knot and its mirror image. Later work has led to other polynomial invariants, namely, the HOMFLY and Kauffman polynomials, and to a more abstract approach to algebraic invariants (Vassiliev invariants and Lie algebras). There are also invariants of different nature: among these, we mention the fundamental group $\pi_1(S^3/K)$ of the knot complement and its hyperbolic volume $v(K)$. The classification of knots and links has led to the important study of braids: these are given by a set of n interlaced strings, with ends defined on two parallel planes, placed at some distance h apart. According to specific topological characteristics, we may consider special types of knot sub-families, such as torus knots, alternating knots, two-bridge knots, tangles, and many others (see Hoste et al., 1998).

Virtual Knots

Virtual knots arise from dynamical flows, generated by the vector field of a specific ordinary differential equation (Ghrist, 1997), in connection with phase-space dynamics and statistical mechanical models (Millett & Sumners, 1994) or, as recently done, from application of ideas from the quantum field theory with an appropriate Lagrangian. This latter approach, originated in work by E. Witten in 1989, has led to the creation of a new area, called the topological quantum field theory, that has proven to be extremely fruitful in providing new results on invariants of low-dimensional manifolds. Soliton knots are given by solutions to soliton equations for one-dimensional systems: in this context there are intriguing questions relating topological invariants, integrability, and conservation laws. Virtual knots and links are generated in visiometrics by numerical simulations: in this case, smooth knots are replaced by polygonal knots, where the number of segments (or sticks) is the result of numerical discretization. Stimulating questions address the minimum number of sticks of given length for each knot type and the generation of knots and links by minimal random walks. Other questions regard charged knots: these are knots and links charged by potentials that generate self-attraction or repulsion on the knot strands (see Figure 3). Under volume-preserving diffeomorphisms, the knot is led to relax by

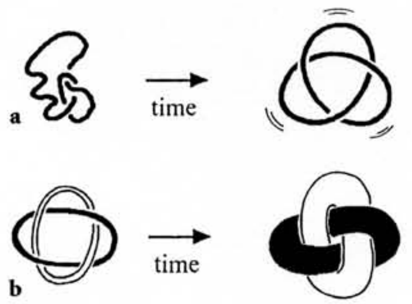

Figure 3. Examples where a topological barrier prevents further relaxation under a volume-preserving diffeomorphism: (a) an electrically charged trefoil knot is maximally extended by the Coulomb repulsion forces to its minimum energy state; (b) a magnetic link attains ground state energy by the action of the Lorentz force on the magnetic volume.

minimizing the knot energy (defined by an appropriate functional), by shrinking or extending the length as far as possible, depending on the potential, to attain an ideal shape (Stasiak et al., 1998). Questions relating to topology and geometry of ideal shapes, and uniqueness of minimum energy states, pose challenging problems at the crossroads of topology, differential geometry, functional analysis, and numerical simulation.

Physical Knots

By physical knots we mean tube models, centered around the knot K, with length $L(K)$, tubular neighborhood of radius $r(K)$, and volume $V(K)$. The tube is filled by vector field lines, whose distribution gives physical properties in terms, for example, of elasticity, vorticity, or magnetic field. A wide variety of filamentary systems present in nature at very different scales can be modeled by physical knots: from DNA molecules, polymer chains, vortex filaments, to elastic cords, strings, and magnetic flux tubes. In fluid systems, the action at a microscopic level of physical processes, such as viscosity and resistivity, may imply changes in knot topology by local recombination of the knot strands (known as knot surgery) and consequential rearrangement of energy distribution. In elastic systems, the material breaking point and internal critical twist are strongly influenced by knot strength and rope length, the latter given by the ratio L/r. All these systems are free to relax their internal energy to states of equilibrium: lower bounds on equilibrium energy for given measures of topological complexity (based, for example, on crossing number information) can be expressed by relationships of the kind

$$E_{\min} \geq h(c, \Phi, V, n), \quad (5)$$

where $E_{\min}$ is the equilibrium energy and $h(\cdot)$ gives the relationship between physical quantities—such as flux Φ, number of components n, knot volume V—and topology, given here by the crossing number c.

Understanding the interplay between topology and energy localization and redistribution can be very important in many fields of science and applications.

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See also **Differential geometry; Dynamical systems; Structural complexity; Topology**

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KOCH CURVE

See **Fractals**

KOLMOGOROV CASCADE

The velocity fluctuations of a high Reynolds number flow in a three-dimensional velocity field are typically dispersed over all possible wavelengths of the system, from the smallest scales, where viscosity dominates the advection and dissipates the energy of fluid motion, to the effective size of the system. This is not so bizarre: our everyday experience tells us it is so. On the corner of a city street, one might watch the fluttering and whirling of a discarded tram ticket as it is swept by an updraught, driven by localized thermal gradients from traffic or air-conditioning units; later, on the television news, one might see reports or predictions of storms on the city or district scale, and a weather map with isobars spanning whole continents. If you are a sailor you will know how to sail, or not, the multi-scaled surface of a turbulent ocean (Figure 1). The mechanism for this dispersal is vortex stretching and tilting: a conservative process whereby interactions between vorticity and velocity gradients create smaller and smaller eddies with amplified vorticity, until viscosity takes over (Tennekes & Lumley, 1972; Chorin, 1994).

An alternative, crude but picturesque, description of multi-scale turbulence was offered by the early 20th century meteorologist Lewis Fry Richardson (1922) in an evocative piece of doggerel: "big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity." Richardson's often-quoted rhyme is apparently a parody of Irish satirist Jonathan Swift's verse: "So, naturalists observe, a flea—Has smaller fleas that on him prey—And these have smaller still to bite'—And so proceed *ad infinitum*."

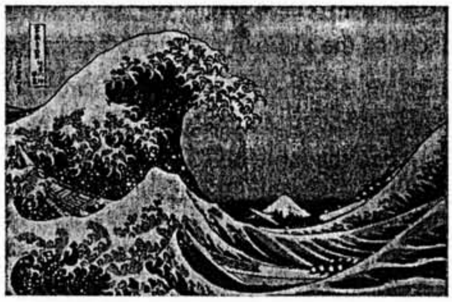

Figure 1. Turbulent action on many different scales in a high Reynolds number flow: woodcut print by Katsushika Hokusai (1760–1849).

The statistics of the velocity fluctuation distribution in turbulent flows were quantified rather more elegantly and rigorously by the mathematician Andrei N. Kolmogorov (1941b), who derived the subsequently famous "$-5/3$ law" for the energy spectrum of the intermediate scales, or inertial scale subrange, of high Reynolds number flows which are ideally homogeneous (or statistically invariant under translation) and isotropic (or statistically invariant under rotation and reflection) in three velocity dimensions. Two thorough, but different in style and emphasis, accounts of Kolmogorov's turbulence work are Monin & Yaglom (1971) and Frisch (1995).

Kolmogorov's idea was that the velocity fluctuations in the inertial subrange are independent of initial and boundary conditions (i.e., they have no memory of the effects of anisotropic excitation at smaller wave numbers). The turbulent motions in this subrange, therefore, show universal statistics, and the flow is self-similar. From this premise Kolmogorov proposed the first hypothesis of similarity as: "For the locally isotropic turbulence the [velocity fluctuation] distributions F_n are uniquely determined by the quantities ν, the kinematic viscosity, and ε, the rate of average dispersion of energy per unit mass [energy flux]." His second hypothesis of similarity is: "For pulsations [velocity fluctuations] of intermediate orders where the length scale is large compared with the scale of the finest pulsations, whose energy is directly dispersed into heat due to viscosity, the distribution laws F_n are uniquely determined by F and do not depend on ν."

Kolmogorov derived the form of the distribution or energy spectrum, which we denote as $\mathcal{E}(k)$, where k is the wave number given by $k^2 = k_x^2 + k_y^2 + k_z^2$, over the inertial subrange simply by dimensional analysis. By the first and second hypotheses, the spectrum must