Complexity measures of tangled vortex filaments

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Abstract We introduce and test measures of geometric and topological complexity to quantify morphological aspects of a tangle of vortex filaments. The tangle is produced by standard numerical simulation of superfluid turbulence in Helium II. Complexity measures such as linking number, writhing number, average crossing number and helicity are computed, and their relation to the energy of the fluid is investigated.

We found a complexity measure -
It really is quite a treasure -
For a vortex entangled,
By methods new-fangled;
I'll explain if you have enough leisure.

1. Introduction

Complex systems of filaments occur frequently in nature. Examples range from vortex structures to magnetic flux tubes to polymers, proteins and DNA. We would like to relate the morphological complexity of such systems with physical properties, such as energy. The aim of this paper is twofold. First we introduce candidate measures of geometric and topological complexity; secondly, we choose superfluid turbulence as a convenient benchmark, compute these measures and compare them to energy.

2. Vortex dynamics and superfluid turbulence

Superfluid turbulence (Barenghi 2001) consists of a disordered, apparently random tangle of vortex filaments. This state of turbulence is particularly simple if compared to traditional hydrodynamics turbulence. Firstly, the superfluid is inviscid. Secondly, all vortex filaments

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have the same circulation Γ (the quantum of circulation). Thirdly, the vortex core radius is so small ($\approx 10^{-8}$cm) and the filaments are so long that the classical theory of thin-core vortex filaments applies well. Unlike what happens in classical turbulence, in which eddies can be of any size and strength, superfluid vorticity is always geometrically well defined (it is the location where both the real and the imaginary parts of the quantum mechanical wave function vanish). All these features make superfluid turbulence a convenient benchmark to study issues of complexity.

Figure 1. Tangle of superfluid vortex filaments

A superfluid vortex line can be described as a closed curve (Schwarz 1988) $\mathbf{X} = \mathbf{X}(s,t)$ where s is arc length and t is time. The line interacts with the thermal excitations present in Helium II, which can be modelled as a viscous fluid of velocity field $\mathbf{v}_n$; the interaction depends on a temperature dependent friction coefficient α. The instantaneous velocity $\mathbf{v}_L$ of a point $\mathbf{X}$ of the superfluid vortex line is given by

$$\mathbf{v}_L = \mathbf{v}_{\text{self}} + \alpha\,\hat{\mathbf{t}} \times (\mathbf{v}_n - \mathbf{v}_{\text{self}}) \ , \tag{1}$$

where $\hat{\mathbf{t}} = d\mathbf{X}/ds$ is the tangent unit vector and the self-induced velocity $\mathbf{v}_{\text{self}}$ is given by the classical Biot-Savart integral

$$\mathbf{v}_{\text{self}}(\mathbf{X}) = \frac{\Gamma}{4\pi} \int_{\mathcal{T}} \frac{\hat{\mathbf{t}} \times (\mathbf{X}^* - \mathbf{X})}{|\mathbf{X}^* - \mathbf{X}|^3}\, ds \ , \tag{2}$$

where $\mathbf{X}^*$ varies along the line and the integral extends to the collection of vortex lines $\mathcal{L}_i$ $(i = 1,..N)$ which form the turbulent tangle $\mathcal{T} = \bigcup_i \mathcal{L}_i$. Equations (1) and (2) are used to determine numerically the time evolution of an initial system of vortex lines in the presence of a given $\mathbf{v}_n$. The computer code also performs vortex reconnections when two vortex lines become very close to each other. In superfluid turbulence reconnections do not involve dissipation and arise from the underlying quantum mechanics in a way which is well described by the Gross - Pitaevskii equation for a Bose - Einstein condensate (Koplik and Levine 1993, Leadbeater et al 2001).

In most experiments (Maurer and Tabeling 1998, Stalp et al 1999) the normal fluid is turbulent. To represent $\mathbf{v}_n$ we perform calculations using two different models. The first is a steady ABC flow (Barenghi et al 1997) given by $\mathbf{v}_n(\mathbf{x}) = (v_{nx}, v_{ny}, v_{nz})$ where $v_{nx} = A\sin(2\pi kz) + C\cos(2\pi ky)$, $v_{ny} = B\sin(2\pi kx) + A\cos(2\pi kz)$ and $v_{nz} = C\sin(2\pi ky) + B\cos(2\pi kx)$ and A, B, C and k are parameters. The second is a more realistic kinematic simulation of turbulence (Kivotides, Barenghi and Samuels 2001) for which

$$\mathbf{v}_n(\mathbf{x}, t) = \sum_{j=1}^{j=J}[\mathbf{A}_j \times \hat{\mathbf{k}}_j \cos(\mathbf{k}_j \cdot \mathbf{x} + \omega_j t) + \mathbf{B}_j \times \hat{\mathbf{k}}_j \sin(\mathbf{k}_j \cdot \mathbf{x} + \omega_j t)] \quad (3)$$

where J is the number of modes used, $\hat{\mathbf{k}}_j$ is a random unit vector, $\mathbf{k}_j = k_j\hat{\mathbf{k}}_j$, $\omega_j = (k_j^3 E_j)^{1/2}$, and the directions and orientations of $\mathbf{A}_j$ and $\mathbf{B}_j$ are chosen randomly but so that the energy of the mode j has the $k_j^{-5/3}$ dependence of Kolmogorov turbulence.

A typical calculation starts with few seeding vortex rings as initial condition (our results do not depend on the initial condition). The initial vortex lines interact with each other and with the background normal fluid, which feeds energy into them. Soon the initial lines become distorted, grow, reconnect, and a vortex tangle is created (see Figure 1).

3. Complexity measures

To analyse the tangle's complexity we project it othogonally onto a given plane. The vortex loops are naturally oriented by the direction of the vorticity, so, using standard convention, we assign an algebraic value $\epsilon_r = \pm 1$ to each apparent point r of self-intersection of the projected tangle. An algebraic measure of the complexity of two loops $\mathcal{L}_i$ and $\mathcal{L}_j$ is the average crossing number (Freedman and He 1991, Moffatt and Ricca 1992)

$$\bar{C}_{ij} = \frac{1}{4\pi}\oint_{\mathcal{L}_i}\oint_{\mathcal{L}_j} \frac{|(\mathbf{X}_i - \mathbf{X}_j) \cdot d\mathbf{X}_i \times d\mathbf{X}_j|}{|\mathbf{X}_i - \mathbf{X}_j|^3} = < \sum_{r \in \mathcal{L}_i \cap \mathcal{L}_j} |\epsilon_r| > , \quad (4)$$

where $\mathbf{X}_i \in \mathcal{L}_i$ and $\mathbf{X}_j \in \mathcal{L}_j$ and the angular brackets denote averaging over all directions $\boldsymbol{\nu}$ of projection. The generalisation to the entire collection of filaments is

$$\bar{C} = \sum_{\mathcal{L}_i,\mathcal{L}_j \in \mathcal{T}} \bar{C}_{ij} \ . \tag{5}$$

For computational simplicity we take only the projections onto the three principal planes ($x = 0$, $y = 0$, $z = 0$) and obtain the estimated average crossing number

$$\bar{C}_\perp = < \sum_{r \in \mathcal{T}_\nu} |\epsilon_r| >_\perp \ , \tag{6}$$

where $< \cdot >_\perp$ indicates that the solid angle average is replaced by the algebraic mean over the three principal planes of projection.

Another interesting quantity is the writhing number Wr which measures the average total coiling of a loop. For a single filament one can show that (Moffatt and Ricca 1992)

$$Wr_i = < \sum_{r \in \mathcal{L}_i \cap \mathcal{L}_i} \epsilon_r > \ , \tag{7}$$

where the average is extended to the number r of apparent, *signed* self-intersections of $\mathcal{L}_i$. The generalisation to the tangle is $Wr = < \sum_{r \in \mathcal{T}} \epsilon_r >$. Again, it is computationally convenient to approximate Wr by the estimated writhing number $Wr_\perp$, given by

$$Wr_\perp = < \sum_{r \in \mathcal{T}_\nu} \epsilon_r >_\perp \ . \tag{8}$$

The linking number Lk_{ij} between two closed loops $\mathcal{L}_i$ and $\mathcal{L}_j$ provides a measure of the topological linking and can be defined as

$$Lk_{ij} = \frac{1}{2} \sum_{r \in \mathcal{L}_i \cap \mathcal{L}_j} \epsilon_r \ . \tag{9}$$

Unlike previous measures, the linking number is a topological invariant because it does not change under continuous deformations of the vortex strands performed by a sequence of Reidemeister moves (Adams 1994), so it is independent of the projection used.

The total linking Lk_{tot} of a system of vortex lines can be defined by

$$Lk_{\text{tot}} = \sum_{\substack{\mathcal{L}_i,\mathcal{L}_j \in \mathcal{T} \\ i \neq j}} |Lk_{ij}| \ , \tag{10}$$

where we deliberately exclude contributions from self-linking (due to writhe and twist of each vortex filament).

The final measure of complexity which we consider is the kinetic helicity (Moffatt 1969) defined by

$$H = \int_V \mathbf{v}_L \cdot \boldsymbol{\omega}\, d^3\mathbf{X} \;, \tag{11}$$

where $\boldsymbol{\omega}$ is the vorticity and the integral is taken over the tangle volume $V = V(\mathcal{T})$. Since vorticity is confined only to vortex lines, $\boldsymbol{\omega}$ is a delta function of strength Γ in the direction $\hat{\mathbf{t}}$ along each filament of $\mathcal{T}$, and we have

$$H = \Gamma \int_{\mathcal{T}} \mathbf{v}_L \cdot \hat{\mathbf{t}}\, ds \;, \tag{12}$$

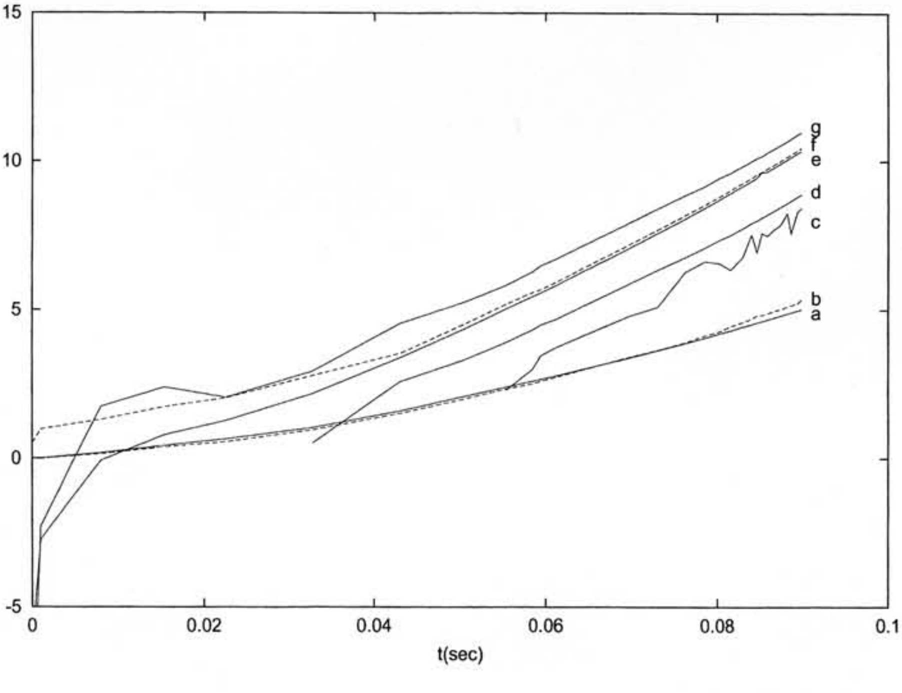

Figure 2. Complexity measures versus time. (a) normalised length $\ln(L/L_0)$; (b) normalised energy $\ln(E/E_0)$; (c) total linking $\ln(Lk_{\mathrm{tot}})$; (d) estimated writhing number $\ln(Wr_\perp)$; (e) average crossing number $\ln(\bar{C})$; (f) estimated average crossing number $\ln(\bar{C}_\perp)$; (g) absolute helicity $\ln(|H|)$.

4. Results

We compute complexity measures as function of time as the vortex tangle grows, and compare one another and against the tangle's total length L and kinetic energy E. Typical results are shown in Figure 2 in the case of an ABC flow (Barenghi, Ricca and Samuels 2001) but essentially similar results are obtained when the Kolmogorov turbulence model is used. The initial rate of growth of L is in agreement with linear stability calculations. As the vortex structure unfolds, the dynamics develops structural complexity and entanglement. It is apparent that the length is a good measure of the energy of the system, as in the trivial case of a single straight vortex line. It is also apparent that, after the initial transient, the growth rate of all complexity measures is essentially the same, and is approximately twice the rate of growth of the length. This means that vortex tangle grows by curling up and folding upon itself more than by spreading and diffusing in space.

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