Energy, helicity and crossing number relations for complex flows

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Abstract Algebraic and topological measures based on crossing number relations provide bounds on energy and helicity of ideal fluid flows and can be used to quantify morphological complexity of tangles of magnetic and vortex tubes. In the case of volume-preserving flows we discuss new results useful to determine lower bounds on magnetic energy in terms of topological crossing number and average spacing of the physical system. New relationships between average crossing number, energy and helicity are derived also for homogeneous vortex tangles. These results find interesting applications in the study of possible connections between energy and complexity of structured flows.

Topological arguments show
That the energy's bounded below;
But what's so engrossing's
The number of crossings,
From which my new insights will flow.

1. Magnetic and vortex knots as standard embeddings

Consider an incompressible and perfectly conducting fluid in an unbounded domain $\mathcal{D}$ of $\mathbb{R}^3$ that is simply connected, with fluid velocity $\boldsymbol{u} = \boldsymbol{u}(\mathbf{x}, t)$, smooth function of the position vector $\mathbf{x}$ and time t and such that $\nabla \cdot \boldsymbol{u} = 0$ in $\mathcal{D}$ and $\boldsymbol{u} = 0$ at infinity. Consider the class of magnetic fields $\{\mathbf{B}\}$ that are solenoidal and frozen in $\mathcal{D}$, that is

$$\mathbf{B} \in \{\nabla \cdot \mathbf{B} = 0 \quad \text{and} \quad \partial_t \mathbf{B} = \nabla \times (\boldsymbol{u} \times \mathbf{B})\} \,. \tag{1}$$

Under ideal conditions these fields have their prescribed topology conserved during time evolution. We restrict our attention to fields that are localised in space and that are indeed confined to tubular neighbourhoods of knots and links. By construction the field is standardly embedded into nested tori T_i centred on smooth loops C_i that are knot-

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ted and linked in $\mathcal{D}$ (Ricca 1998). For simplicity we take field lines that are closed on themselves within each tube and that have same flux Φ. We therefore identify an n-component magnetic link with the standard embedding of a disjoint union of n magnetic solitori in $\mathcal{D}$:

$$\sqcup_i T_i \quad (i = 1, \ldots, n) \hookrightarrow L_m := \mathrm{supp}(\mathbf{B}) \ . \tag{2}$$

Similarly, by replacing the magnetic field $\mathbf{B}$ with the vorticity field $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$, chosen so as to satisfy (1) and (2), we obtain vortex knots and links L_v.

Let us denote by $V = V(L_m)$ the total volume of the magnetic link and consider the evolution of L_m under the action of the group of volume- and flux-preserving diffeomorphisms

$$\varphi : \ L_m \to L_{m,\varphi} := \varphi_* L_m \ . \tag{3}$$

The magnetic energy and the helicity of L_m are defined by

$$E(L_m) \equiv \int_V \|\mathbf{B}\|^2 \, d^3\mathbf{x} \ , \quad H(L_m) \equiv \int_V \mathbf{A} \cdot \mathbf{B} \, d^3\mathbf{x} \ , \tag{4}$$

where $\mathbf{A}$ is the vector potential associated with $\mathbf{B} = \nabla \times \mathbf{A}$ ($\nabla \cdot \mathbf{A} = 0$). An important issue in topological fluid mechanics is to relate geometric and topological properties to the energy and helicity of the fluid system (for an introductory review see Ricca & Berger 1996).

2. Helicity, linking and average crossing numbers

It is well known that helicity admits topological interpretation in terms of linking numbers. We have:

Theorem (Moffatt 1969; Berger & Field 1984; Moffatt & Ricca 1992): *Let L_m be a collection of magnetic links (knots). Then*

$$H(L_m) = \sum_i Lk_i \, \Phi_i^2 + 2 \sum_{i \neq j} Lk_{ij} \, \Phi_i \Phi_j \ , \tag{5}$$

where Lk_i denotes the (Călugăreanu-White) linking number of C_i with respect to the framing induced by the embedding of the **B***-field in T_i, and Lk_{ij} denotes the (Gauss) linking number of T_i with T_j.*

The Gauss linking number $Lk_{ij} = Lk(C_i, C_j)$ between C_i and C_j is a fundamental topological invariant of link types and it is obviously conserved under frozen field evolution. The linking number admits interpretation in terms of minimal number of crossings of the link type: by

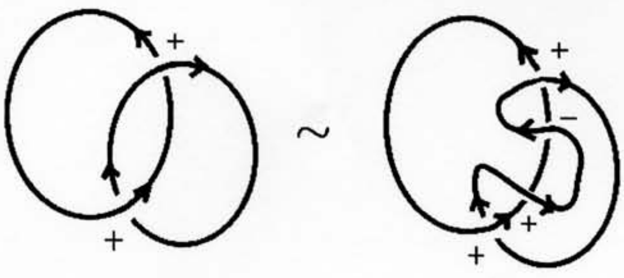

Figure 1. The Gauss linking number is a topological invariant of the link type and its calculation, based on the sum of the signed crossings, is independent of the projected diagram. The case $Lk = +1$ is shown here.

assigning $\epsilon_r = \pm 1$ to each crossing present in a given projected diagram of the oriented link (omitting self-crossings and following a standard sign convention), it can be expressed as sum over r signed crossings, according to the formula

$$Lk_{ij} \equiv \int_{ij} d\omega = \frac{1}{2} \sum_{r \in \mathcal{C}_i \sqcap \mathcal{C}_j} \epsilon_r \qquad (i \neq j) \; , \tag{6}$$

where $d\omega$ is the classical Gauss integrand form and $\sqcap$ denotes over- and under–crossings in plane projections (omitting self–crossings; see Figure 1). The Călugăreanu-White linking number Lk_i is also a topological invariant associated with the field embedded in T_i and it can be decomposed in two geometric quantities

$$Lk_i = Wr_i + Tw_i \; , \tag{7}$$

the writhing number Wr_i (which measures the average space coiling of $\mathcal{C}_i$ in $\mathcal{D}$) and the total twist Tw_i present in T_i.

Linking numbers are closely related to the average crossing number $\bar{C}$, which is an algebraic measure of the link complexity in space (Freedman & He 1991). For any pair i and j of components this quantity is simply given by the average total number of apparent crossings present in the link and is defined by

$$\bar{C}_{ij} \equiv \int_{ij} |d\omega| = < \sum_{r \in \mathcal{C}_i \# \mathcal{C}_j} |\epsilon_r| > \; , \tag{8}$$

where $\#$ denotes all possible crossings (including self-crossings) of the curves and $< \cdot >$ average over all possible planes. Now, since

$$\left| \int_{ij} d\omega \right| \leq \int_{ij} |d\omega| \; , \tag{9}$$

and holds true for any given pair of components i and j, the inequality can be extended to the whole system, hence by using (6) and (8), we have

$$|2\sum_{i\neq j} Lk_{ij}| = |2\sum_{i\neq j}(\frac{1}{2}\sum_{r\in\mathcal{C}_i\cap\mathcal{C}_j}\epsilon_r)| \leq \sum_{ij}\sum_{r\in\mathcal{C}_i\cap\mathcal{C}_j}|\epsilon_r| \leq \sum_{ij}\bar{C}_{ij} \equiv \bar{C} \ . \quad (10)$$

3. Topological bounds on magnetic energy

We consider the magnetic relaxation of the magnetic link L_m subject to a volume- and flux–preserving diffeomorphism, as discussed by Moffatt (1992). For simplicity we assume that all link components have same flux Φ and that are all zero–framed, that means $Lk_i = 0$ for each $i = 1, \ldots, n$. An interesting result that relates magnetic energy and complexity of the physical system is given by the following:

Theorem (Freedman & He 1991): *Let L_m be an essential magnetic link (or knot), then we have*

$$E(L_m) \geq \left(\frac{16}{\pi}\right)^{1/3} \frac{\Phi^2 \bar{C}}{V(L_m)^{1/3}} \ . \quad (11)$$

Note that $\bar{C} \geq C_{\min}$, where $C_{\min}$ denotes the topological crossing number of the knot or link type. Moreover Moffatt has shown (1992) that the energy is bounded from below, according to the inequality

$$E(L_m) \geq q_0\,|H(L_m)| \ , \quad (12)$$

where $q_0 > 0$ depends on the geometry of supp($\mathbf{B}$), with a spectrum of ground states given (1990) by

$$E_{\inf}(L_m) = m\,\Phi^2 V(L_m)^{-1/3} \ , \quad (13)$$

where m is related to the topology of the system. The problem whether these infima can actually reach their minimum value remains open. Both q_0 and m were left undetermined.

By using the result of Freedman & He, combined with the inequality (10), we can show (for a detailed discussion, see Ricca 2002) that

$$\mathrm{i}): \ q_0 = \left(\frac{16}{\pi V(L_m)}\right)^{1/3}, \quad \text{and} \quad \mathrm{ii}): \ m = \left(\frac{16}{\pi}\right)^{1/3} C_{\min} \ . \quad (14)$$

Equation (14–i) and (12) state that for given linking complexity (with notable exceptions: see Figure 2b), the smaller the volume, the higher the energy level. This is in agreement with the intuitive idea that for a

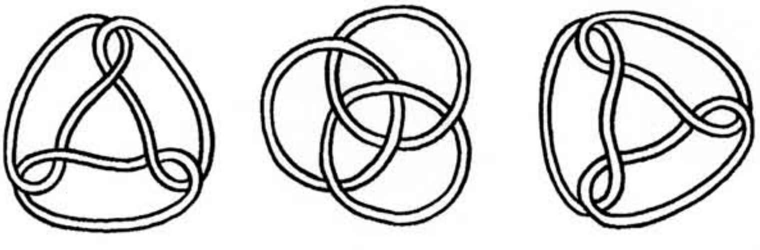

Figure 2. Three topologically distinct link types with $n = 3$ and $C_{\min} = 6$: (a) 6_1^3: $Lk = -3$; (b) 6_2^3 (Borromean rings): $Lk = 0$; (c) 6_3^3: $Lk = -1$.

knotted rope the tighter the knot, i.e. the smaller the physical space it occupies, the higher is its potential energy.

Equation (14–ii) identifies m with the topological crossing number of the knot/link type. This result, combined with eq. (13), provides an interesting and powerful relationship between ground state energy and topology and shows the intimate relationship between the two. By direct inspection of the link tabulation, it is however immediately evident that there may be countably many topologically distinct links with equal number n of components and same topological crossing number $C_{\min}$, leaving partially open the fundamental question of identifying uniquely ground state energies with topology. An example is given by Figure 2, where three distinct link types with $n = 3$ and $C_{\min} = 6$ are shown: assuming for these links same volume and flux, then all three links have same $E_{\inf}$.

4. Vortex tangles and complexity

Finally, let us consider a tangle L_v of (zero-framed) vortex filaments in a steady state. As above, all filaments are assumed to have same strength Φ. Kinetic energy T and helicity are defined as usual by

$$T(L_v) \equiv \frac{1}{2}\int_V \|\boldsymbol{u}^2\|\, d^3\mathbf{x} \;, \quad H(L_v) \equiv \int_V \boldsymbol{u}\cdot\boldsymbol{\omega}\, d^3\mathbf{x} \;. \tag{15}$$

Since

$$\left|\int_V \boldsymbol{u}\cdot\boldsymbol{\omega}\, d^3\mathbf{x}\right| \leq \int_V |\boldsymbol{u}\cdot\boldsymbol{\omega}|\, d^3\mathbf{x} = \Phi^2 \int_{ij} |d\omega| \;, \tag{16}$$

by applying Hölder inequality to the second term, we have

$$|H(L_v)| \leq \Phi^2 \bar{C} \leq \sqrt{2K\Omega} \;, \tag{17}$$

where $\Omega = \int_V \|\boldsymbol{\omega}\|^2 \, d^3\mathbf{x}$ is the total enstrophy of the tangle. Work on possible new relationships between complexity, topology and energy of vortex tangles is currently under way and new results on complexity measures can be found in the paper by Barenghi, Ricca & Samuels (2001) (see also the paper in this volume).

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