Tropicity and Complexity Measures for Vortex Tangles

Renzo L. Ricca

[1] Mathematics Department, University College London
Gower Street, London WC1E 6BT, UK
[2] Isaac Newton Institute for Mathematical Sciences
20 Clarkson Road, Cambridge CB3 0EH, UK
ricca@math.ucl.ac.uk

Abstract. In this paper we introduce and discuss new concepts useful to analyse and characterize patterns of vortex lines in fluid flows. We define measures of tropicity to identify 'tubeness', 'sheetness' and 'bulkiness' of vortex lines and to measure the spreading of field lines about preferred directions. Algebraic, geometric and topological measures based on crossing number information are discussed and are put in relation to the kinetic helicity and the energy of the fluid system.

1 Vortex Structures and Tangles in Classical and Quantized Vortex Flows

Coherent structures represent an essential feature of classical turbulent flows [2], [5]. Experimental and numerical results have indeed shown that vorticity has a tendency to coalesce into highly localized regions. As modern visualizations of classical and superfluid flows show [16], [14], [15], [17], strong anisotropies emerge as vortical flows re-organize themselves to form tubes, sheets or quantized complex tangles of vortex lines. As vortex structures evolve, different types of non-linear effects and instabilities take place, until continuous break-up and re-structuring are overcome by total dissipation.

The tremendous progress in visualization techniques and real-time diagnostics of complex flow patterns [18] makes it now possible accurate recognition of vortex pattern formation and interaction; detailed mechanisms of braiding, linking and re-structuring of vortex lines can be analysed by real-time simulations to a high degree of accuracy. In this context geometric and topological information is available from data-sets of numerical simulations; properly analysed, it provides valuable help to understand fundamental properties of dynamics and energetics of turbulent flows [13], [11]. New concepts and tools based on geometry and topology are therefore being developed [12] to quantify physical information associated with structural complexity of such flows. In the following sections we shall introduce measures based on geometric and topological concepts that will provide useful tools to quantify structural complexity of vortical flows and help to develop new measures to estimate physical properties.

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2 Measures of Tropicity for Vortex Tangles: Tubeness, Sheetness and Bulkiness

A first step in the application of measures of structural complexity is to develop tools for pattern recognition of vortex structures. These must be based on estimates of anisotropy and spatial extension of vortex lines in the fluid. Let us consider a fluid region $\mathcal{D}$ in $\mathbb{R}^3$ and a generic tangle $\mathcal{T}$ of n vortex lines $\mathcal{L}$, i.e. $\mathcal{T} = \bigcup_n \mathcal{L}_n$ in $\mathcal{D}$. A vortex line is given by a line of vorticity, whose support is identified with a smooth, simple space curve, not necessarily closed in $\mathcal{D}$ ($\mathcal{D}$ can be a sub-region of the entire fluid domain), with vorticity everywhere tangent to the curve. It is useful to introduce the concept of 'tropicity' (a measure of the space configuration of the vortex region) to characterize the degree of tubeness, sheetness and bulkiness of $\mathcal{T}$ in $\mathcal{D}$. Consider a single vortex line $\mathcal{L}$ as frozen in space and time. A measure of the spatial extent of $\mathcal{L}$ in $\mathcal{D}$ is given by the maximal distance D_1 between two points P_i and P_j on $\mathcal{L}$, i.e.

$$D_1 = \max_{i,j} \mathrm{d}(P_i, P_j) \equiv \overline{P_0 P_1} = \sqrt{X^2 + Y^2 + Z^2}\ , \tag{1}$$

where $P_0 = (x_0, y_0, z_0) \in \mathcal{L}$, $P_1 = (x_1, y_1, z_1) \in \mathcal{L}$ and $X = x_1 - x_0$, $Y = y_1 - y_0$, $Z = z_1 - z_0$. The unit vector $\hat{\mathbf{T}}_1 = (P_1 - P_0)/D_1$ (*first directional tropicity vector*) given by the director cosines X/D_1, Y/D_1, Z/D_1, is the principal directional axis of $\mathcal{L}$. Let us consider then the transversal spatial extension of $\mathcal{L}$. Take the maximal distance of a third point $P_i \in \mathcal{L}$ (not aligned with P_0 and P_1) from the line $l(P_0, P_1)$: a measure of maximal width spanned by $\mathcal{L}$ in $\mathcal{D}$ is given by

$$D_2 = \max_{i} \mathrm{d}(P_i, l(P_0, P_1)) \equiv \overline{P_2 O}\ , \tag{2}$$

where $P_2 \in \mathcal{L}$ and $O \in l(P_0, P_1)$, where O, being not necessarily a point on $\mathcal{L}$, is the footpoint of the orthogonal projection of P_2 onto $\overline{P_0 P_1}$. The *second directional tropicity vector* is the unit vector $\hat{\mathbf{T}}_2 = (P_2 - O)/D_2$, which prescribes direction and orientation of the second principal directional axis of $\mathcal{L}$ in $\mathcal{D}$. We can now define the plane $\Pi = \Pi(P_0, P_1, P_2)$ and, by taking the maximal distance of a fourth point $P_i \in \mathcal{L}$ from Π, we have a measure of the maximal three-dimensional extension of $\mathcal{L}$ in $\mathcal{D}$, i.e.

$$D_3 = \max_{i} \mathrm{d}(P_i, \pi(P_0, P_1, P_2)) \equiv \overline{P_3 Q}\ , \tag{3}$$

($P_3 \in \mathcal{L}$) taken along the direction of the *third directional tropicity vector* $\hat{\mathbf{T}}_3 = \hat{\mathbf{T}}_1 \times \hat{\mathbf{T}}_2$.

Let us consider the whole tangle of vortex lines $\mathcal{T}$. The spatial extension of $\mathcal{T}$ in $\mathcal{D}$ is measured by the three maximal distances D_i, this time taken with respect to the principal axes $\boldsymbol{\lambda}_i$ (along $\hat{\mathbf{T}}_i$, $i = 1, 2, 3$) determined by sampling P_i over the whole tangle $\mathcal{T}$. The tropicity volume is given by $V(\mathcal{D}) = D_1 D_2 D_3$, and measures of tropicity can be defined by taking the relative ratio of these quantities. We have

- if $D_3 = O(D_2)$ and $D_2 \ll D_1$: *tubeness* $\stackrel{\text{def}}{=} \Lambda_1 \equiv \dfrac{D_1}{D_2}$;
- if $D_3 \ll D_2$ and $D_2 = O(D_1)$: *sheetness* $\stackrel{\text{def}}{=} \Lambda_2 \equiv \dfrac{D_2}{D_3}$;
- if $D_3 = O(D_2)$ and $D_2 = O(D_1)$: *bulkiness* $\stackrel{\text{def}}{=} \Lambda_3 \equiv \dfrac{D_1 D_2 D_3}{D_3^3}$.

These quantities provide a first crude information of the space configuration of the pattern. For classical turbulent flows these quantities are particularly useful to detect regions of high tubeness and sheetness, whereas for quantized flows bulkiness may provide more useful information related to vortex line density. Moreover, as we shall see below, information on directional tropicity finds useful applications for geometric and topological estimates of structural complexity.

3 Measures of Geometric Complexity: Directional Alignment and Writhing

In classical fluids strong anisotropy is characterized by the presence of elongated, tubular regions. If $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ denotes vorticity ($\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$ being the velocity field function of the position vector $\boldsymbol{x}$), and $\boldsymbol{\sigma}$ vortex stretching, defined by

$$\sigma_i = S_{i,j}\omega_j = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\omega_j \ , \tag{4}$$

then a measure of how vorticity and vortex stretching lines are spread about the principal directional axis $\hat{\mathbf{T}}_1$ is given by the angles η_ω and η_σ, defined by

$$\tan\eta_\omega = \frac{|\bar{\boldsymbol{\omega}} \times \hat{\mathbf{T}}_1|}{\bar{\boldsymbol{\omega}} \cdot \hat{\mathbf{T}}_1} \ , \qquad \tan\eta_\sigma = \frac{|\bar{\boldsymbol{\sigma}} \times \hat{\mathbf{T}}_1|}{\bar{\boldsymbol{\sigma}} \cdot \hat{\mathbf{T}}_1} \ , \tag{5}$$

where $\bar{\boldsymbol{\omega}} = \langle \boldsymbol{\omega} \rangle_{\mathcal{D}}$ and $\bar{\boldsymbol{\sigma}} = \langle \boldsymbol{\sigma} \rangle_{\mathcal{D}}$ denote space averages of the field lines over the tropicity domain $\mathcal{D}$. Similarly, we can define the spread of vorticity and stretching (through the angles χ_ω and χ_σ) about a sheet-like distribution by taking

$$\cot\chi_\omega = \frac{\bar{\boldsymbol{\omega}} \cdot \hat{\mathbf{T}}_3}{|\bar{\boldsymbol{\omega}} \times \hat{\mathbf{T}}_3|} \ , \qquad \cot\chi_\sigma = \frac{\bar{\boldsymbol{\sigma}} \cdot \hat{\mathbf{T}}_3}{|\bar{\boldsymbol{\sigma}} \times \hat{\mathbf{T}}_3|} \ . \tag{6}$$

For quantized vortex tangles directional writhing provides a measure of average coiling of vortex lines in space. Let $\Pi_i = \Pi(\hat{\mathbf{T}}_i)$ denote a plane of projection with normal $\hat{\mathbf{T}}_i$ and consider the projection of the tangle $\mathcal{T}$ onto the plane Π_i. A 'good' diagram $\mathcal{T}_i = \mathcal{T}(\hat{\mathbf{T}}_i)$ (which of course depends on the line of projection $\hat{\mathbf{T}}_i$) is given by a curve graph, whose self-intersections are given by countably many points, where the segments intersect transversally. Good projections can be found by an appropriate choice of the projection plane. By keeping track of the orientation of the curve (induced by the vorticity vector), we obtain an oriented diagram (see Fig. 1) and by assigning the value $\epsilon_r = \pm 1$ to each projected

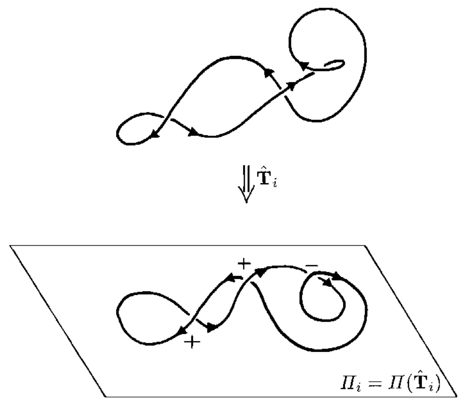

Fig. 1. Example of an oriented space curve projected onto the plane Π_i.

crossing r (according to standard convention on signs [6]), we can quantify the directional writhing in terms of algebraic sum of positive and negative crossings on Π_i (see [10])

$$Wr(\mathcal{T}_i) = \frac{1}{4\pi} \int_{\mathcal{T}_i} \frac{d\boldsymbol{X} \times d\boldsymbol{X}^* \cdot (\boldsymbol{X} - \boldsymbol{X}^*)}{|\boldsymbol{X} - \boldsymbol{X}^*|^3} = \sum_{r \in \mathcal{T}_i} \epsilon_r \ . \tag{7}$$

The writhing number $Wr(\mathcal{T})$ is given by averaging the directional writhe over the whole solid angle. It may be computationally convenient to approximate this quantity by taking the algebraic mean over the three principal orthogonal planes as reference projection planes; hence, the estimated writhing number given by

$$Wr_\perp = \left\langle \sum_{r \in \mathcal{T}_i} \epsilon_r \right\rangle_\perp \approx Wr(\mathcal{T}) \ , \tag{8}$$

will provide a reasonable, simple measure of the average coiling of $\mathcal{T}$.

4 Algebraic Measure of Structural Complexity: Average Crossing Number

Structural complexity can be measured by counting the total number of crossings, that are present in a tangle at a given time. This quantity, which is associated with the un-oriented tangle diagram, is given by the average crossing number $\bar{C}$ by counting the total number of un-signed crossings in $\mathcal{T}_i$ and averaging over the whole domain $\mathcal{D}$ [3]. We have

$$\bar{C}(\mathcal{T}) = \left\langle \sum_{r \in \mathcal{T}_i} |\epsilon_r| \right\rangle_{\mathcal{D}} \ . \tag{9}$$

Once again, it is computationally convenient to approximate this measure by the algebraic mean taken over the three principal orthogonal planes, hence

$$\bar{C}_{\perp} = \left\langle \sum_{r \in \mathcal{T}_i} |\epsilon_r| \right\rangle_{\perp} \approx \bar{C}(\mathcal{T}) \ . \tag{10}$$

Current work done by Barenghi *e*t al., [1] shows that $\bar{C}_{\perp}$ provides indeed a good approximation to $\bar{C}$ and it seems to be very effective to detect structural complexity.

5 Measures of Topological Entanglement: Kinetic Helicity and Directional Linking

Topological entanglement can be calculated on the basis of information about directional tropicity of $\mathcal{T}$. In this context the concept of linking number is fundamental, since it is related to the kinetic helicity H of the flow (see [7])

$$H(\mathcal{T}) = \int_{\mathcal{D}} \boldsymbol{u} \cdot \boldsymbol{\omega} \, dV \ . \tag{11}$$

For superfluid vortices helicity can be estimated through linking number measures made directly on the vortex tangle, hence avoiding the difficulties associated with the integration over the vorticity field. These are based on the concept of (Gauss) linking number $Lk_{ij} = Lk(\mathcal{L}_i, \mathcal{L}_j)$ of two loops $\mathcal{L}_i$ and $\mathcal{L}_j$, given by

$$Lk_{ij} = \frac{1}{4\pi} \oint_{\mathcal{L}_i} \oint_{\mathcal{L}_j} \frac{d\boldsymbol{X}_i \times d\boldsymbol{X}_j \cdot (\boldsymbol{X}_i - \boldsymbol{X}_j)}{|\boldsymbol{X}_i - \boldsymbol{X}_j|^3} \ , \tag{12}$$

and its limit form Lk_{ii}, given by the the Călugăreanu-White formula

$$Lk_{ii} = Wr_i + Tw_i \ , \tag{13}$$

where $Wr_i = Wr(\mathcal{L}_i)$ is the writhing number of $\mathcal{L}_i$ and $Tw_i = T_i + N_i$ is the total twist, sum of the total torsion (T_i) and intrinsic twist (N_i) of $\mathcal{L}_i$ divided by 2π (see [10] for precise definitions and physical meaning of these quantities). For a tangle of n vortex lines, each of circulation κ_i ($i \in [1, \ldots, n]$), the kinetic helicity is given by (see [11])

$$H(\mathcal{T}) = 2 \sum_{i \neq j} Lk_{ij} \, \kappa_i \kappa_j + \sum_i Lk_{ii} \, \kappa_i^2 \ . \tag{14}$$

Note that in the case of superfluid vortices, since quantized vortex lines have no internal structure we can assume $Tw_i = T_i$.

In the case of strong anisotropy of vortex flows directional tropicity provides natural reference directions to measure localized winding of vortex lines. A relative measure of winding is given by the directional linking number $\ell k_{\boldsymbol{\lambda}_i}$ of vortex lines with respect to one of the principal tropicity axes $\boldsymbol{\lambda}_i$, i.e.

$$\ell k_{\boldsymbol{\lambda}_i} = \langle \ell k(\mathcal{L}_i, \boldsymbol{\lambda}_i) \rangle_{\mathcal{D}} \ , \tag{15}$$

keeping $\boldsymbol{\lambda}_i$ fixed in the average process.

6 Relationships Between Complexity Measures and Energy Levels

It is of fundamental importance to relate measures of algebraic, geometric and topological complexity to physical properties of the system, such as kinetic helicity and energy. For this purpose it is convenient to re-write eq. (14) in a more compact form. Consider the linking numbers L_{ij} $(i, j \in [1, \ldots, n])$ as elements of a square matrix; since $L_{ij} = L_{ji}$, we can reduce the linking matrix to diagonal form, i.e.

$$\begin{pmatrix} L_{11} & L_{12} & \ldots & L_{1n} \\ L_{21} & L_{22} & \ldots & L_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ L_{n1} & L_{n2} & \ldots & L_{nn} \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} M_{11} & 0 & \ldots & 0 \\ 0 & M_{22} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & M_{nn} \end{pmatrix} , \tag{16}$$

where each element M_{ii} takes into account self- and mutual linking of the vortex lines. We can therefore re-cast eq. (14) in the form

$$H(\mathcal{T}) = \sum_{i=1,\ldots,n} M_{ii}\, f(\kappa_i) , \tag{17}$$

where $f(\cdot)$ is a linear function of quadratic terms in κ_i.

If the tangle is made on average of vortex filaments of same length L (obtained by an average measure over the tropicity domain $\mathcal{D}$), we can show [9] that on dimensional grounds the enstrophy Ω of the system is given by a relationship of the form

$$\Omega(\mathcal{T}) = \int_{\mathcal{D}} |\boldsymbol{\omega}|^2 \, dV = \frac{1}{L} \sum_{i=1,\ldots,n} M_{ii}\, f(\kappa_i) , \tag{18}$$

that provides an interesting connection with helicity. Moreover, since the magnetic energy of a perfectly conducting magnetized fluid is bounded from below by the magnetic helicity H_m [8], according to the inequality

$$E_{\min} \geq q_0 |H_m| , \tag{19}$$

where q_0 is a positive constant, then we can expect that in steady state conditions similar bounds hold for minimum enstrophy levels or for other types of ground state energy in relation to the complexity of the physical system.

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