Rediscovery of Da Rios equations

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The remarkable story of the discovery of a set of equations at least three times this century shows once again that independent discoveries can occur and exist for some time.

IN the frenetic activity that seems to characterize modern science, the same discovery may well have more than one parent and, sometimes, more than one life. The following story is a remarkable example of this, for it shows not only that genuine independent discoveries may occur and even today survive independently for some time, but also provides grounds to reflect on common attitudes in science as well as insight into the way in which our scientific culture is formed.

The story begins at the University of Padua in the early years of this century. There, a young Italian, Luigi Sante Da Rios, under the supervision of Tullio Levi-Civita, one of the great mathematicians of this century, had just completed the work for his *Laurea* in mathematics, the Italian degree roughly equivalent to a master of science. The subject of Da Rios' research was fluid dynamics from the point of view of applied mathematics, in particular the study of the free motion of a slender tube-like shaped vortex (a vortex filament) in an unbounded gaseous or liquid fluid medium.

The problems of vortex motion and dynamics of rotational flows had captured imagination since the time of ancient Greece, and throughout the eighteenth and nineteenth centuries many efforts had been made to give these studies rigorous mathematical foundation. At the beginning of this century, then, the vivid influence of works like those done by Lord Kelvin and von Helmholtz upon mathematicians interested in vorticity dynamics was already widely acknowledged. And it was from these studies that Da Rios started to investigate the dynamics of an isolated thin vortex filament as an object embedded in an infinite domain, entirely filled by a homogeneous, incompressible, inviscid fluid. He wanted to study the influence that the localized vorticity has on the local behaviour of the vortex; to analyse the kinematics and the dynamics of the vortex filament so as to understand its motion; and, eventually, to apply these results to examine its global behaviour.

By assuming the thinness of the vortex core and the local effects due to vorticity as relevant hypotheses to his mathematical model, Da Rios derived the related asymptotic potential theory to find a simple approximate expression for the self-induced velocity of the vortex. This law says that the vortex filament moves with a velocity proportional to its local curvature: if we take two vortex rings (a smoke ring is a good example) of equal internal structure and equal core size, but of different diameter, then the smaller ring will move faster. The set of basic assumptions that leads to this law of motion is nowadays commonly referred to as the localized induction approximation (LIA). Using LIA, Da Rios went further and studied the global behaviour of the space vortex filament. He found a remarkable system of two coupled equations, hereafter referred to as the Da Rios equations, which governs the motion of the vortex in terms of time evolution of its geometric intrinsic parameters, curvature and torsion. He completed his research by demonstrating the existence of two physical solutions to these equations describing rigid motion in the intrinsic parameters. One solution describes the configuration of a plane vortex that moves with a 'catenary'-like shape, a shape taken under the gravitational force by a flexible, inextensible thread of uniform weight suspended at two points. The other describes the configuration of a space vortex moving with a particular three-dimensional shape like a travelling wave.

All these results were collected by Da Rios, then 25 years old, in a paper[1] entitled "*Sul moto d'un liquido indefinito con un filetto vorticoso di forma qualunque*" (On the motion of an unbounded fluid with a vortex filament of any shape), printed in 1906 in the July–August issue of *Rendiconti del Circolo Matematico di Palermo*, an international mathematical journal in which, at that time, mathematicians like Cantor, Hilbert, Poincaré, Peano and many others used to publish. Unfortunately, the paper didn't bring much recognition to Da Rios. Only Levi-Civita, who already knew the results, gave the author the merit due to him. In a paper published in 1908[2], Levi-Civita, re-examining the whole work on the asymptotic theory from another point of view, states: "For a long time in hydrodynamics, in the study of rectilinear or circular vortex filaments, we had to deal with special asymptotic expressions. But Da Rios has the merit of being the first to establish a general approach of this kind" (the general asymptotic theory and the LIA velocity). At that time Levi-Civita was interested in the gravitational attraction produced by a slender filament of mass for possible applications to the saturnian ring, and he worked for several years on improvements of the asymptotic potential theory and on its applications[3–5].

In the meantime, Da Rios became assistant lecturer at the University of Padua; there, very probably, the daily discussions he would have had with Levi-Civita, who is known to have been

SUL MOTO D'UN LIQUIDO INDEFINITO CON UN FILETTO VORTICOSO

che, per le (18) e (21), diventano:

$$-\frac{d\tau}{dt}-\left(\frac{c''}{c}-\tau^2\right)'=cc',$$

$$c''=c''-c\tau^2+c\tau^2,$$

$$\frac{dc}{dt}=c\tau'+2c'\tau.$$

Abbiamo quindi finalmente le equazioni cercate:

$$(22)\qquad \begin{cases} \dfrac{dc}{dt}=c\tau'+2c'\tau, \\ \dfrac{d\tau}{dt}=-cc'+\left(\tau^2-\dfrac{c''}{c}\right)'. \end{cases}$$

Il teorema di esistenza, applicato a questo sistema di equazioni a-mette di asserire con tutto rigore che le funzioni $c(s, t)$, $\tau(s, t)$ regolarità) univocamente definite dai valori inizi

The intrinsic equations (22) as they were presented by Da Rios in his first paper published in 1906; *c* and τ stand for curvature and torsion of the vortex filament, respectively.

enthusiastic about his work, convinced him to publish his results once again. He prepared two versions[6,7]: one, the shorter, was presented by Levi-Civita himself for publication, and published in the *Rendiconti della Reale Accademia dei Lincei*; the other, written as a review article, appeared in 1910, in the spring issue of *Rendiconti del Circolo Matematico di Palermo*. In both papers, Da Rios explained once again in detail the asymptotic potential theory he had developed, the LIA approach and the intrinsic equations of motion. Still, for some unknown reason, both works remained obscure, and Da Rios turned his attention to other aspects of vortex dynamics. Levi-Civita, on the other hand, was apparently so fascinated by the Da Rios results that in 1932 he collected them in a survey[8], improving and extending the Da Rios analysis in many directions. He studied in detail the stability of the circular vortex, the case of a helicoidal vortex filaments and a particular plane configuration, characterized by a knot-like shape, for a vortex moving without change of form; recalling what Da Rios had briefly sketched, he analysed in the intrinsic parameters the theory of rigid vortex motion.

Many years passed, and this last work of Levi-Civita also fell into obscurity. To understand this lack of interest it is useful to remember that at the beginning of the century, and for several decades thereafter, the problems considered fundamental in fluid dynamics were the structure and the role of the boundary layer produced by a fluid flowing along a solid surface, the formation of wakes behind obstacles, and the study of the fully developed turbulence. It is not surprising, then, that more than 50 years passed before the localized induction concept was rediscovered, when Hama and coauthors[9-11], unaware of Da Rios's work, proposed the LIA approach in several papers published in the well-known US journal *Physics of Fluids*. It is common to refer to Hama as having formulated LIA; nonetheless in 1962 he acknowledged Arms for having conceived the LIA concept[10].

An immediate consequence was the independent rediscovery of the Da Rios equations in fluid dynamics, in 1965, by Betchov[12], who also found the rigid plane vortex solution originally derived by Da Rios in 1906 and later studied by Levi-Civita. A few years later, by coupling curvature and torsion of the filament into one complex variable, Hasimoto[13] combined the two intrinsic equations (then termed the Betchov equations) into the nonlinear Schrödinger equation, one of the 'soliton' equations now much studied in the mathematics of nonlinear systems, such as optics, acoustics and signal-transmission theory. Soliton equations give travelling wave-like solutions: on a space curve, for example, soliton solutions are three-dimensional pulse-like solitary waves travelling on the curve, asymptotically preserving their shape and velocity upon collision with other solitary waves. Under LIA, one of these travelling waves is given by the three-dimensional physical solution found by Da Rios.

Sometimes theories about different aspects of physics may not differ much in their mathematical structure; exact or formal analogies can occur. Analogies are useful in science, because if we deal with the same mathematical model applied to different physical contexts, and are looking in the same direction, we are likely to find the same things, even if we call them by different names. This is true in the case of Lakshmanan and his colleagues[14], who studied, in the context of ferromagnetic media, the dynamics of a one-dimensional system of classical spins — that is a space curve on which the spin vectors, in the continuum limit, lie. Under certain assumptions it is possible to show that the motion of such an object is governed according to LIA. By taking this into consideration, Lakshmanan and colleagues derived, for the third time, the Da Rios equations and found the solitary wave solutions. What is surprising, though, is that when Lakshmanan presented these equations again in a brief letter in 1977[15], he mentioned Betchov and Hasimoto but did not reference the equations that Betchov had earlier (re)discovered.

A whole family of travelling wave solutions, initially investigated by Levi-Civita in his paper of 1932, were found successfully by Kida[16] in 1981, and the soliton behaviour of the related nonlinear Schrödinger equation began to be well known. Of the soliton behaviour of his intrinsic equations Da Rios was of course unaware; nonetheless when he studied the three-dimensional pulse-like travelling-wave solution he entitled that part of his paper "Research of hunched vortices travelling rigidly".

It was only at the beginning of the 1980s that Germano became aware of Da Rios' work and wrote a paper[17] in which he reaffirmed the discoveries of Da Rios and extended the original set of intrinsic equations by considering the full general kinematics of a space curve, independently of the particular physical context. This paper, presented in 1983 at an Italian congress, was translated into English, but unfortunately this version was never published. Da Rios became widely known only when Moffatt[18], informed by Germano about his work, acknowledged him in his talk at a symposium in 1983 held in Kyoto, this time in front of an international audience and, of course, given in English!

But surprises are still possible. Recently, Hama[19] gave his account of the genesis of LIA: talking about his 'theorem', he said that according to Hasimoto, LIA was formulated in Tokyo in 1937[20]. And Keener[21] last year derived again the set of general intrinsic equations, as Germano had done in 1983, acknowledging Betchov and Germano only in a note added in proof.

With the growth of interest in exotic string objects — classical, cosmic, super or whatever — and with the application of the mathematics of nonlinear systems in many fields of physics, chemical physics and biological physics, attention given to the Da Rios–Betchov equations has increased greatly. Although connections between the intrinsic kinematics of curves and the general properties of one-dimensional soliton equations have been brought into evidence[22], many questions remain open. Recently, further generalizations of the intrinsic equations to higher-dimensional cases have been proposed[23], which may be interesting not only because the underlying mathematical structure of the equations could provide further evidence of links to more general sets of non-linear equations, but also because many physical problems might find in this approach their suitable formulation. In pure mathematics, too, these equations can be of interest: various problems in the geometry of string-generated surfaces[24], await solution. My own feeling is that the story of the Da Rios equations is not yet finished. □

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