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Quantized Vortex Knots

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We have investigated numerically the motion and the stability of quantized vortex knots.

Turbulence in superfluid helium II manifests itself as a tangle of quantized vortices.[1] The study of the tangle was initially motivated by engineering applications of heat transfer and by the desire to understand a peculiar form of disorder in the vicinity of absolute zero. The more recent interest comes from the study of isothermal helium turbulence and its possible connections with classical turbulence,[2-4] and from experiments in which a tangle of vortex lines is interpreted as a model of cosmic strings in the early universe.[5]

Since the core structure is very small (of the order of the coherence length, $\xi \approx 10^{-8}$ cm) and dissipative effects are negligible at sufficiently low temperatures, superfluid vortices can be well approximated by vortex line singularities which move in an ideal Euler fluid. The motivation behind this work is the remark that, in the absence of dissipation and reconnections, the topological properties of these singularities are preserved during the time evolution. Vortices which are initially knotted or linked may evolve into very complex structures, but will preserve the kind of knot or link which ties them together. Although dissipation and reconnections will eventually alter the topology, the lifetime of these structures may be sufficiently long to be significant in transporting properties throughout the fluid and mediating different stages of the flow's evolution.[6]

More in general, it is important to understand the topological properties of the vortex structures which are possible under the laws of vortex dynamics. It is known that the motion of a vortex line can be determined from its

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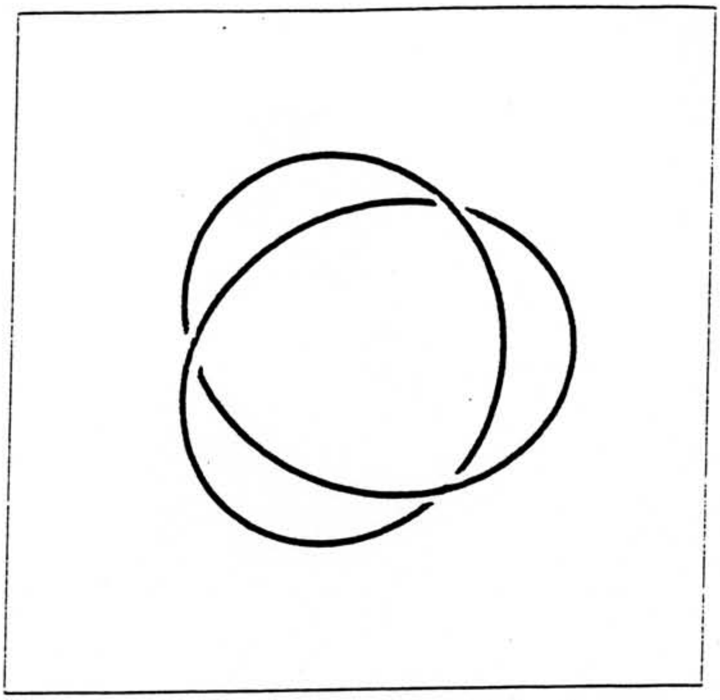

Fig. 1. The vortex knot $T_{2,3}$ seen from the top. Broken lines represent where one vortex line crosses underneath another.

own shape via the Biot - Savart law[7] (BSL). Unfortunately the BSL involves an integration along the entire vortex configuration, which hinders analytic progress and makes numerical calculations very expensive. The introduction of the local induction approximation[8] (LIA) to the BSL alleviates this difficulty and sets a well - defined, more tractable mathematical problem.

The interest in the solutions which are possible under BSL and LIA motivates recent applications of topological ideas to fluid dynamics and magneto - hydrodynamics. Coherent structures which have the form of knotted tubes and links occur often in nature beyond the realm of superfluidity. Examples range from eddies and vortex tubes in classical turbulence,[9-12] to tornados in geophysical fluid dynamics,[13] to plasma loops in solar magneto - hydrodynamics.[14] All these structures can be modeled using the theory of vortex filaments.

The aim of this letter is to present some preliminary results which involve a single vortex line which is wrapped many times around a doughnut to form a non - intersecting closed curve called a *torus knot*[15,16] $T_{p,q}$. Here $p > 1$ and $q > 1$ are co - prime integers. The first integer, p, tells how many times the vortex line wraps around the torus in the direction of the large radius, R_L, of the torus; the second integer, q, defines the number of wraps in the direction of the small radius, R_S. The ratio $w = q/p$ is called the winding number. An example of torus knot is shown in figure 1. A torus knot is thus a vortex configuration of non trivial topological complexity, intermediate between a simple vortex ring and a fully developed vortex tangle.

We have studied numerically the motion and the stability of some vortex knots. Calculations have been performed both under the LIA, to make contact with analytic theory, and under the full BSL, to obtain exact results. The algorithm consists in subdividing a given initial vortex knot into a number N of small segments. The position $\mathbf{s}$ of each segment was then integrated in time t according to either the full BSL

$$\frac{d\mathbf{s}}{dt} = \frac{\Gamma}{4\pi} \int \frac{(\mathbf{z} - \mathbf{s}) \times d\mathbf{z}}{|\mathbf{z} - \mathbf{s}|^3}, \tag{1}$$

or the LIA

$$\frac{d\mathbf{s}}{dt} = \beta \mathbf{s}' \times \mathbf{s}'' \tag{2}$$

where Γ is the circulation around the vortex line, a prime denotes derivative with respect to arclength, $\beta = (\Gamma/4\pi)\log(R/a_0)$, and R and a_0 are respectively the local radius of curvature and the vortex core parameter. The technique to carry out this calculation, which involves adjustments of N during the evolution, and the tests which have been performed, has already been described in the literature,[17-19] and would take too long to repeat here. Our work differs from the studies of vortex tangles done by Schwarz,[17] Samuels[18] and Aarts and DeWaele[19] in that we neglect dissipation, that is to say there is no mutual friction[20,21] in our model. This is equivalent to a vortex system at a rather low temperature .

The first result of our numerical investigations is that we have confirmed the stability criterium of Ricca,[15] who predicted that small amplitude vortex knots are stable under the LIA if $w > 1$. For example we have been able to integrate the motion of $T_{2,3}$ for a distance over 40 times the large radius, over which the knot retained its shape as expected (see figure 2). The knot $T_{3,2}$, on the contrary, became unstable immediately, as was predicted by Ricca. We must qualify the use of the term "stable" in this context. We include no dissipation in our calculation. With no dissipation to damp out small perturbations, the best that can be achieved is neutral stability. Over time, these perturbations will build up and the vortex filament will drift away from the original configuration. This drift is visible in the long-time evolution of the "stable" knot configurations in figures 2 and 3.

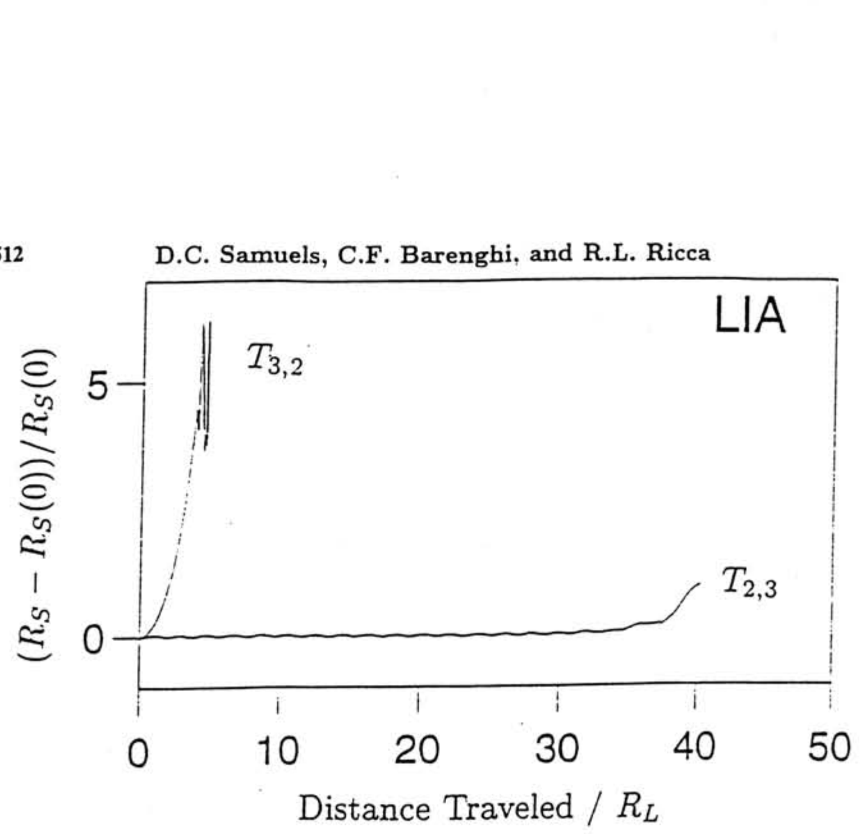

Fig. 2. The evolution of the vortex knots $T_{2,3}$ and $T_{3,2}$ under LIA. Plotted is the measured small radius R_S of the toroidal vortex knot (minus the original small radius $R_S(0)$) as a function of the distance traveled by the knot.

The second result which we report is the surprising finding that *both* $T_{2,3}$ and $T_{3,2}$ evolve in a stable way if the exact BSL is used to integrate the time evolution, instead of the LIA. Figure 3 shows the results for a $T_{3,2}$ vortex knot moving under LIA, where the vortex knot is unstable, and under BSL. The motion under the exact BSL is stable. Again, since we use no dissipation in these simulations, the knot is only neutrally stable, but it is clear that this same initial condition has a very different motion under the LIA and the BSL, and it survives for a much longer time under the exact BSL. Thus, while LIA allows analytic results to be developed for vortex motion these results may be quite different from the actual motion of the vortex.

The results presented here shed new light on the study of vortex structures with nontrivial topology. The exact Biot-Savart law is shown to give significantly different motion of these vortices than the simple Local Induction Approximation. Work is in progress to study other knotted vortex structures, including searching for structures which undergo reconnections and maintain shape and stability.

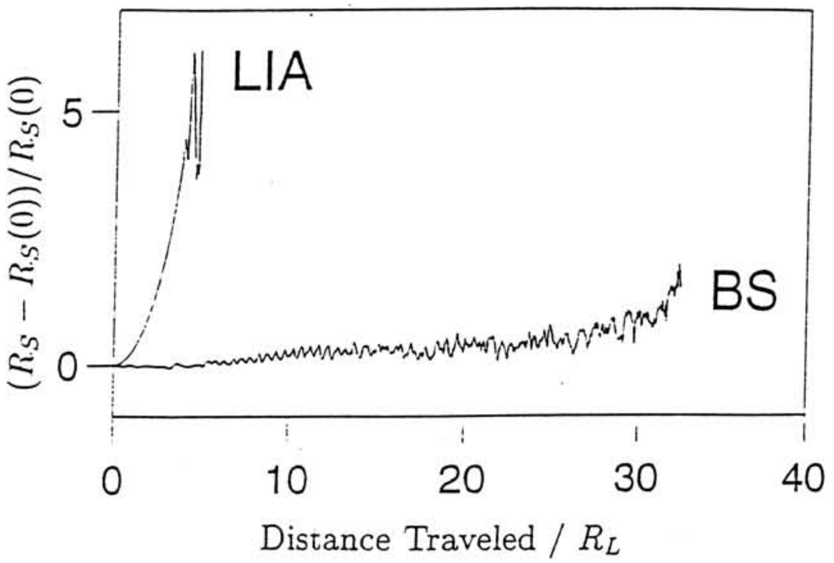

Fig. 3. The evolution of the vortex knot $T_{3,2}$ under LIA and BS. The axes are as in figure 2.

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