The contributions of Da Rios and Levi-Civita to asymptotic potential theory and vortex filament dynamics

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Abstract

In this paper we present for the first time a detailed account of the work of L.S. Da Rios and T. Levi-Civita on what is believed to be one of the first major contributions to three-dimensional vortex filament dynamics. Their work spanned a period of almost 30 years, from 1906 to 1933, and despite many publications remained almost unnoticed throughout this century. After a partial re-discovery (Ricca, 1991a), new material has now been found and is presented here with a full review of their work in relation to the present state of the art in non-linear mechanics and vortex dynamics. Their results include the conception of the localized induction approximation (LIA) for the induced velocity of thin vortex filaments, the derivation of the intrinsic equations of motion, the asymptotic potential theory applied to vortex tubes, the derivation of stationary solutions in the shape of helical vortices and loop-generated vortex configurations and the stability analysis of circular vortex filaments. In the light of modern developments in non-linear fluid mechanics, their work strikes for modernity and depth of results. Even more striking is the fact that this work remained obscure for almost a century. The results of Da Rios are particularly important in the study of integrable one-dimensional systems and vortex filament motion; Levi-Civita's work on asymptotic potential for slender tubes is at the core of the mathematical formulation of potential theory and capacity theory.

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1. Introduction

In this paper we present for the first time a detailed account of the work of L.S. Da Rios and T. Levi-Civita on what is believed to be one of the first major contributions to the study of three-dimensional vortex filament motion. We discuss the origin of the localized induction approximation (LIA) concept for the induced velocity of thin vortex filaments in ideal fluids and the intrinsic equations of motion derived by Da Rios. We review the work of Levi-Civita on the asymptotic potential

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for slender tubes and application to vortex motion. The results of Da Rios on stationary vortex solutions, which include helical and "pseudo-helical" solutions and a class of loop-generated vortex configurations, are also presented and discussed for the first time after their original publication.

The relevance of their work and its implication in vortex dynamics and in the more general context of modern non-linear mechanics is impressive. The extraordinary aspect of their work is that despite their effort to publish the results, their discoveries remained obscure throughout this century. Gradually their work has been independently re-discovered by a plethora of post-war authors (in particular Arms and Hama, 1965; Betchov, 1965; Batchelor, 1967) and is still subject of active research, from the study of one-dimensional integrable systems and solitons (see, for example, Sym, 1984; Langer and Perline, 1991; Nakayama et al., 1992; Ricca, 1993, 1995) to vortex dynamics and asymptotic theory (see, for example, Kida, 1981; Keener, 1990; Ricca, 1992, 1994). With this paper we want to pay tribute to their work by presenting a review of their contribution in the light of the present state of the art in vortex filament dynamics and non-linear mechanics.

The paper is organized as follows. In Section 2 we give an account of the scientific life of Da Rios and his relationship with Levi-Civita. Some information about the social and cultural aspects of their lives is included to help understand better Da Rios's work and put it into historical perspective. In Section 3 we discuss the origin of the localized induction approximation (LIA) concept. The intrinsic equations for the motion of a vortex filament are derived in Section 4. The work of Levi-Civita on the asymptotic form of the Newtonian potential applied to vortex filament dynamics is reviewed in Section 5. In Section 6 we present Da Rios's stationary solutions in the shape of helical vortex filaments and loop-generated vortex configurations and discuss the stability analysis of circular vortex filaments, carried out by Levi-Civita. Final remarks are drawn in Section 7. Whenever possible we tried to retain the same notation as in the original papers.

2. The scientific life of Da Rios and his relationship with Levi-Civita

Son of Giuseppe Da Rios, a carpenter, and Benvenuta Dall'Ava, a country girl, Luigi Sante Da Rios was born in Santa Lucia di Piave, a small village in the province of Treviso (Italy), on 2 April 1881. After completion of his secondary studies as a private student at the Liceo Ginnasio Canova, where he received classical education, in 1902 he enrolled in the University of Padua for a four-year course in mathematics. At that time the faculty boasted of a number of outstanding mathematicians and scientists: the father of the absolute differential calculus G. Ricci-Curbastro, the astronomer G. Lorenzoni, the geometer G. Veronese and one of their former students, the brilliant 29 year-old T. Levi-Civita. From the university scripts we can see that Da Rios was quite successful in both pure and applied subjects, especially in Rational Mechanics and Higher Mechanics, two taught courses given by Levi-Civita. This is also documented by the file of his successful annual applications for studentships and fee exemptions (Fig. 1a). Influenced by Levi-Civita's own passion for fluid mechanics, at the age of 23 he became interested in vortex dynamics and started to work on the motion of vortex filaments in perfect fluids.

We should remember that until then very little was known on the evolution of vortex structures. The mathematical foundation of vortex theory was laid down not long before by Helmholtz (1858) and, with the exception of Kelvin's works on vortex rings (1867, 1869) and columnar vortices (1880), not much was known about fully three-dimensional structures. The lack of results concerning

Fig. 1. (a) Letter of Da Rios to the Rector of The University of Padua asking for the exemption of the annual university fees, dated 15 November 1905. On the right-hand side of the letter are the marks for three third-year exams: Rational Mechanics (Levi-Civita) 30/30; Astronomy (Lorenzoni) 28/30; Higher Mechanics (Levi-Civita) 30/30 *"cum laude"*. Immediately below, the fee exemption granted and signed on 21 March 1906. (b) Luigi Sante Da Rios (photograph dated 14 July 1909).

b)

Fig. 1. Continued

the evolution of three-dimensional vortex filaments must have stimulated the curiosity of the young Da Rios. The mathematical difficulties, however, were not negligible. The major problem was represented by the asymptotic evaluation of the Biot-Savart integral, which gives the induced velocity of the vortex. This integral diverges when the field point approaches the source point on the filament (with the typical singularity associated with the Green function), so that the analytical evaluation of the velocity becomes difficult.

By assuming the thinness of the vortex core and the local effects due to vorticity as relevant hypotheses to his mathematical model, Da Rios developed an asymptotic theory (now known as localized induction approximation, or LIA, for short), which allowed him to analyse the evolution of thin vortex filaments in intrinsic geometric terms (intrinsic equations). This research, which was part of his thesis, became a substantial piece of work and was published in the *Rendiconti del Circolo Matematico di Palermo* in May 1906, just a few months before his graduation (9 July 1906).

The development of an asymptotic theory for local induction and the intrinsic geometric study are in many ways a result of the very lively scientific atmosphere of the time. Padua was then the centre of an unrivaled Italian school in differential geometry, with people like Bianchi, Cesàro, Ricci and Levi-Civita. So much that the *Rendiconti*, founded in 1884, attracted immediately mathematicians like Darboux, Hilbert, Poincaré, Klein, and, of course, the Italians, including Bianchi and Levi-Civita (see the historical account by Nastasi, 1988). Despite such a favourable atmosphere, the paper of Da Rios remained unnoticed.

After graduation, Da Rios became assistant lecturer at the University of Padua, where he taught algebra and calculus for a few years (1907–1911). During this time Levi-Civita developed an asymptotic theory for the Newtonian potential for slender tubes (with possible applications to Saturnian rings and vortices; 1908a, b, 1912) and encouraged Da Rios to publish abridged versions of his original work (1909, 1910) and some new results (1911, 1916a, b, c). In these papers, Da Rios explained once again the local asymptotic theory he had developed and the intrinsic equations of motion. He also analysed fluid motion in the interior of vortex tubes and the existence of stationary helical solutions. This further publicity brought his work some recognition: A.E.H. Love quoted the intrinsic equations in his article for the *Encyclopédie des Sciences Mathématiques Pures et Appliquées* (IV-18, pp. 143–144, 1912) (we also know that part of his results were discussed in N.S. Vasil'ev's thesis, examined by Joukowskii in 1913), and he received a prize of 2000 liras by the Ministry of Education for a competition of the *Accademia dei Lincei* (31 December 1912).

In a paper of 1908, Levi-Civita, talking about the origin of the localized induction approximation, states: "For a long time in hydrodynamics, in the study of rectilinear or circular vortex filaments, we had to deal with special asymptotic expressions. Da Rios has the merit to be the first to have established a general approach of this kind." Still, for some unknown reason, his results remained little known. For example, in 1912 Cisotti was invited to review the progresses done in hydrodynamics by himself, Levi-Civita and his co-workers at a national convention in Rome. Among several works quoted (1913) there were many, including two of his own, published in the *Rendiconti* in 1906–1908, but, astonishingly, he made no reference to Da Rios, who, for instance, attended the congress with Levi-Civita. Only years later (1922) Cisotti publicly acknowledged Da Rios's results at an international meeting in Innsbruck, in a memoir published in the conference proceedings edited by von Karman and Levi-Civita (1924). There is little doubt, though, that the Italian academic establishment of the early 20s could not be aware of Da Rios's work.

After a second degree in civil engineering (12 December 1913), Da Rios developed an interest in applications. Family pressure (he married and had five children) and social difficulties (Italy entered World War I in 1915) pushed him to look for more practical applications of his studies and he became more interested in engineering (he wrote more than 30 papers between 1912 and 1940). In 1917 he conceived and patented a device, called "ring propeller" (*elica ad anello*), made of a propeller with a ring casing mounted around it to improve performance, and from 1916 onwards he carried out several experiments on his device.

During this time Levi-Civita kept an interest in his pupil's work and career, and in 1927 Da Rios was awarded a second prize of 5000 liras. The prize was given explicitly for the invention of the ring-propeller and the experimental work but, as we can see from the transcript of the prize committee report, there is also clear appreciation for his first papers. The report is signed by Levi-Civita, F. Severi, O.M. Corbino (the then director of the Physics Institute in Rome) and N. Parravano. From 1921 to 1937 Da Rios took part in six competitions for a university chair (in Messina, Cagliari and Catania) without success. In 1930 a series of experimental tests on his device were carried out at the aeronautical laboratories of the Politecnico di Torino, then under the direction of L. Panetti, and the results were disastrous.

In the meantime Levi-Civita completed his work on the asymptotic potential theory, with new results on vortex filament motion. Invited by H. Villat in Paris, Levi-Civita collected and presented his results with those of Da Rios in a series of lectures, in March 1931 (1932a, b). This gave Da Rios a chance to review his original work on vortex filaments (1930), and derive new stationary solutions

in the shape of helical and loop-generated vortex configurations (1931a, b, c, 1933a, b). Unfortunately, neither the publication of Villat's text book on vortex theory (1930) nor a German thesis on the Italian contributions to vortex dynamics (Wehner, 1939) had the chance to quote these works.

The dictatorship in Italy made things very difficult for the Jews and Levi-Civita was no exception. Oppressed by the fascism, he died in Rome on 20 December 1941, leaving behind a flourishing school of mathematicians. Da Rios, after his unsuccessful attempts in university, became headmaster of the Liceo Ginnasio "M. Foscarini", a grammar school in Venice, until retirement (1951). The bitterness for the lack of greater acknowledgement for his mathematical and experimental work remained hidden in a very reserved and austere character. In the forties, some applications of his invention brought him short-lived hopes. Despite some recognition for his services in education and many publications, he became disillusioned and entered a long and deep depression. He eventually died in Padua on 10 October 1965. In the same year Arms and Hama re-derived the LIA and Betchov re-discovered his original intrinsic equations.

3. Origin of the localized induction approximation

The localized induction approximation (LIA) concept is due to Da Rios and was developed in the first eight sections of his paper of 1906 (hereafter denoted by DR1).

Consider an incompressible, inviscid fluid in an infinite domain $\mathscr{D}$ of $\mathbb{R}^3$. Let (x, y, z) be the Cartesian coordinates of a point P in $\mathscr{D}$ and denote with t time and $v = (v_1, v_2, v_3)$ the velocity of a fluid particle at P. Vorticity is defined by

$$\omega = \nabla \times v \quad \text{(DR1, Eq. (1))} \tag{1}$$

(in DR1 we have $(v_1, v_2, v_3) \equiv (u, v, w)$ and $\omega = 2(p, q, r)$), with $\nabla \cdot v = 0$ in $\mathscr{D}$ and $v = 0$ at infinity. Let us consider an isolated vortex filament (closed on itself or extending to infinity) in $\mathscr{D}$, with a circular cross-section of radius $a \ll R$ (where R is the radius of curvature of the vortex centreline) and circulation κ ($\kappa = 2\omega$ in DR1). Let us identify the thin filament with the vortex line $\mathscr{L}$, smooth and free from self-intersection. Let O be an arbitrary point on $\mathscr{L}$, origin of the (right-handed) intrinsic Frenet frame (t, n, b), given by the unit tangent, normal and binormal vectors to $\mathscr{L}$ at O (note that in DR1 this triad is chosen to coincide with a *left-handed* Cartesian reference centred at O; see Fig. 2). Vorticity is given simply by $\omega = \omega t$. On $\mathscr{L}$ take a point Q, of coordinates (ξ, η, ζ) smooth functions of arc-length s (with $s = 0$ at O), sufficiently close to O. The velocity induced by the vortex line $\mathscr{L}$ at an external point P is given by the Biot-Savart integral (in the left-handed frame)

$$v = \frac{\kappa}{4\pi} \int_{\mathscr{L}} \frac{t_Q \times (Q - P)}{r^3} \, ds \quad \text{(DR1, Eq. (5))}, \tag{2}$$

where $t_Q = (d\xi/ds, d\eta/ds, d\zeta/ds)$ is the unit tangent at Q, and $r = \overline{PQ}$. By Taylor's expansion up to terms of third order, we have

$$\begin{aligned}
\xi(s) &= s + s^3 \phi, \\
\eta(s) &= (cs^2/2) + s^3 \psi, \quad \text{(DR1, Eq. (7))}, \\
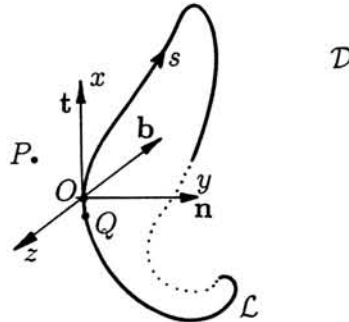\zeta(s) &= s^3 \chi,
\end{aligned} \tag{3}$$

Fig. 2. The vortex filament is identified with the vortex line $\mathscr{L}$. $Q = (\xi, \eta, \zeta)$ is a point on $\mathscr{L}$, and $P = P(x, y, z)$ is an external point in the fluid domain. The intrinsic reference frame is centred at the origin O on $\mathscr{L}$.

where c is the local curvature and ϕ, ψ, χ are smooth functions of s, continuous and finite in the neighbourhood of O.

We want to evaluate (2) at neighbouring points, and to do this we consider how field points near the vortex filament approach source points on $\mathscr{L}$. Let $\varepsilon = |P - O|$ and α, β, γ be the direction cosines of ε, i.e.

$$x = \varepsilon\alpha, \quad y = \varepsilon\beta, \quad z = \varepsilon\gamma. \tag{4}$$

We have

$$r^2 = (\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2 = \Delta^2 + \mathscr{P}s^3, \tag{5}$$

where

$$\Delta^2 = (1 - c\varepsilon\beta)s^2 - 2\varepsilon\alpha s + \varepsilon^2, \tag{6}$$

and where $\mathscr{P} = \mathscr{P}(s, \varepsilon, \alpha, \beta, \gamma)$ is a continuous and finite function of the arguments. It is more convenient to refer to the (independent) variables $|s|$ and ε using the polar coordinates (ρ, ς); hence,

$$|s| = \rho \cos \varsigma, \quad \varepsilon = \rho \sin \varsigma. \tag{7}$$

Let us consider the conical surface K with vertex at O, axis along the tangent t at O, and small but finite angle of semi-aperture $\delta_0 = O(a/R)$ (see Fig. 3a). The surface K sub-divides $\mathscr{D}$ into a conical region containing $\mathscr{L}$, and a region exterior to K. Without loss of generality we assume that a neighbouring point P approaches O only from the region exterior to K, with angle of incidence $\delta > \delta_0$ (hence, we say that P tends to O "non-axially"). Under this condition, we can prove (DR1, pp. 121–123) that

$$\left| \frac{1}{r^3} - \frac{1}{\Delta^3} \right| \leqslant \frac{\Upsilon}{\rho} \quad \text{(DR1, Eq. (15))}, \tag{8}$$

with $\Upsilon > 0$ constant, and use this inequality to evaluate (2).

Let us now decompose $\mathscr{L}$ into two parts (Fig. 3b): one, centred at O and of finite length $2l$ ($l \ll L$), is given by the arc QOQ'; the other, denoted by Λ, is the remaining part complement to
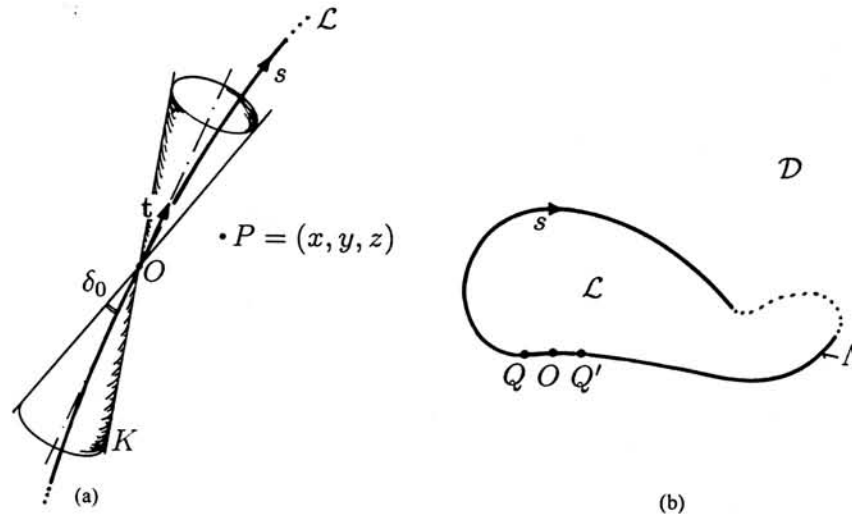
Fig. 3. (a) P can approach the origin from the fluid sub-domain exterior to the conical surface K of small and finite semi-aperture δ_0. (b) The vortex line $\mathcal{L}$ can be sub-divided into two parts: one, of length $2l$ (arc QOQ') and centred at the origin O, and the remaining part Λ, complement to $\mathcal{L}$.

$\mathcal{L}$. The velocity at P, v_P, can thus be decomposed into two contributions: $v^{(a)}$, due to the induction effects of the arc QOQ', and $v^{(f)}$, due to Λ, i.e.

$$v = v^{(a)} + v^{(f)} \quad \text{(DR1, Eq. (5'))}. \tag{9}$$

Note that the contribution $v^{(f)}$ is finite, even when P is asymptotically near to O. Here we are primarily interested in the asymptotic behaviour of the function $v^{(a)}$ as $P \to O$ (in the sense specified above); therefore, let us evaluate the Biot-Savart integral (2) according to the decomposition (9). By (3), (4), (7) and the inequality (8), we have (DR1, pp. 123–125)

$$v_1 = \frac{\kappa}{4\pi} \int_{-l}^{l} \frac{c\varepsilon\gamma s}{\Delta^3} \, ds + v_1^{(f)}, \tag{10}$$

$$v_2 = -\frac{\kappa}{4\pi} \int_{-l}^{l} \frac{\varepsilon\gamma}{\Delta^3} \, ds + v_2^{(f)}, \tag{11}$$

$$v_3 = \frac{\kappa}{4\pi} \int_{-l}^{l} \frac{\frac{1}{2}cs^2 - \varepsilon\alpha cs + \varepsilon\beta}{\Delta^3} \, ds + v_3^{(f)}, \tag{12}$$

so that $v^{(f)} = (v_1^{(f)}, v_2^{(f)}, v_3^{(f)})$ remains finite as $P \to O$. Since we are interested in the asymptotic behaviour of the induced velocity, we shall ignore finite contributions. By using (6), (8) and asymptotic analysis, we can see that the first integral on the right-hand side of (10) (denoted by A, in Eq. (16) of DR1) gives $v_1^{(a)} = 0$; so that ignoring finite terms (and dropping the superscript), we write

$$v_1 = 0. \tag{13}$$

Similarly, inspection of (11) and (12) reveals that both integrals depend only on ε and the local geometry of $\mathscr{L}$. Ignoring finite contributions, we have

$$v_2 = -\frac{\omega\gamma}{\varepsilon\pi(1-\alpha^2)}, \quad v_3 = \frac{\omega\beta}{\varepsilon\pi(1-\alpha^2)} - \frac{\omega c}{2\pi}\log\varepsilon. \tag{14}$$

Hence, the asymptotic velocity contributions along the tangent, normal and binormal directions (denoted here by v_t, v_n, v_b, with $\kappa = 2\omega$) are given by

$$v_t = 0, \quad v_n = 0, \quad v_b = \frac{\kappa c}{4\pi}\log\varepsilon \tag{15}$$

(cf. Eq. (17) of DR1, where the minus sign is due to the opposite orientation of the binormal unit vector). Now, by replacing $\log\varepsilon$ (which varies slowly) with a constant (in the third of Eq. (15)) and after re-scaling, Eqs. (15) reduce to

$$v = cb, \tag{16}$$

which is the well-known form given by the localized induction approximation (LIA) (neglecting finite contributions due to distant parts and self-interaction). Hence, under LIA vortex filaments move simply in the binormal direction with speed proportional to the local curvature.

4. Intrinsic equations

The intrinsic equations governing vortex motion were derived by Da Rios in the second part of his work (DR1, pp. 129–133). These equations are given in terms of time derivatives of curvature and torsion of $\mathscr{L}$. In this paper we derive them as a particular case of a more general and straightforward approach (Germano, 1983; Ricca, 1991b).

Let $X = X(s,t)$ denote the vortex line $\mathscr{L}$ and $X' = t$ the unit tangent vector (primes denote arc-length derivatives). Tangent, normal and binormal satisfy the Serret–Frenet relations

$$t' = cn, \quad n' = -ct + \tau b, \quad b' = -\tau n, \tag{17}$$

where τ denotes torsion. First, let us consider the problem of filament motion from a purely kinematical viewpoint. Let us write the velocity as

$$\dot{X} = v = v_t t + v_n n + v_b b, \tag{18}$$

where everything is a smooth function of s and t (overdots denote time derivatives). We have

$$\dot{t} = \dot{X}' = v', \tag{19}$$

which by Eq. (18) gives

$$\dot{t} = At + Bn + Cb, \tag{20}$$

where

$$A = v_t' - cv_n, \quad B = cv_t + v_n' - \tau v_b, \quad C = \tau v_n + v_b'. \tag{21}$$

By the Frenet-Serret equations (17), we have

$$\dot{t}' = \dot{c}n + c\dot{n},$$ (22)

$$\dot{n}' = -(\dot{c}t + c\dot{t}) + \dot{\tau}b + \tau\dot{b},$$ (23)

and by equating the arc-length derivative of Eqs. (19) to (22), we have

$$\dot{n} = Dt + En + Fb,$$ (24)

where

$$cD = A' - cB, \quad cE = cA + B' - \tau C - \dot{c}, \quad cF = \tau B + C'.$$ (25)

Similarly, by equating the arc-length derivative of Eqs. (24) to (23), we have

$$\dot{b} = Gt + Hn + Kb,$$ (26)

where

$$\tau G = D' - c(E - A) + \dot{c}, \quad \tau H = c(D + B) + E' - \tau F, \quad \tau K = \tau E + F' + cC - \dot{\tau}.$$ (27)

Moreover, since

$$\dot{t} \cdot t = \dot{n} \cdot n = \dot{b} \cdot b,$$ (28)

$$\dot{t} \cdot n + t \cdot \dot{n} = \dot{n} \cdot b + n \cdot \dot{b} = \dot{b} \cdot t + b \cdot \dot{t} = 0,$$ (29)

by Eqs. (20), (24) and (26), we have

$$A = E = K = 0, \quad D = -B, \quad G = -C, \quad H = -F.$$ (30)

By the first of Eqs. (21) and (30), we have the relation

$$v'_t = cv_n,$$ (31)

which means simply that $\mathcal{L}$ is arc-length parametrized. The other two conditions $E = 0$ and $K = 0$ give the intrinsic equations for $\mathcal{L}$, i.e.

$$\dot{c} = (cv_t + v'_n - \tau v_b)' - (\tau v_n + v'_b)\tau,$$ (32)

$$\dot{\tau} = \left[\frac{(cv_t + v'_n - \tau v_b)\tau + (\tau v_n + v'_b)'}{c}\right]' + (\tau v_n + v'_b)c.$$ (33)

Under LIA $v_t = v_n = 0$ and $v_b = c$, so that Eqs. (32) and (33) become (DR1, Eqs. (22))

$$\dot{c} = -c\tau' - 2c'\tau,$$ (34)

$$\dot{\tau} = \left(\frac{c''}{c} - \tau^2\right)' + cc',$$ (35)

which are the intrinsic equations originally derived by Da Rios (the difference in sign is due to the opposite binormal orientation: cf. Eq. (18) of DR1, p. 129). These equations prescribe (up to rigid motion) the evolution of the vortex filament in $\mathcal{D}$ for given initial conditions $c(s, 0)$ and $\tau(s, 0)$.

5. Formulation of the asymptotic potential theory and application to vortex filament dynamics

The study of the asymptotic form of the Newtonian potential for points very near to a tube source was developed by Levi-Civita in a number of papers from 1908 to 1932. It stemmed from the original work of Da Rios on the localized induction approximation and was applied to various physical problems, including the induction due to electric currents in a wire (1909), the gravitational effect associated with Saturnian rings (1912, 1932a) and vortex motion (1932a). The asymptotic theory was re-organized by Levi-Civita for his Paris lectures of March 1931. Here we review this work, based on the series of published lectures (Levi-Civita, 1932a; here-after denoted by LC), where the asymptotic potential theory finds application to vortex filament motion.

5.1. Asymptotic expressions for line potential, Newtonian potential of slender tubes and gradient

We begin by considering the elementary case of a segment OH of length l and homogeneous, constant, linear density v_0. Let O be the origin of a reference Ox, and let Q denote a generic point of OH at x (see Fig. 4a). The Newtonian potential U associated with OH and evaluated at an external point M is defined by

$$U = v_0 \int_0^l \frac{\mathrm{d}x}{r_1},$$ (36)

where $r_1 = \overline{QM}$. If $\varepsilon = \overline{OM}$ and $\psi_1 = \widehat{MOH}$, then

$$r_1^2 = (x - \varepsilon \cos \psi_1)^2 + \varepsilon^2 \sin^2 \psi_1.$$ (37)

A straightforward calculation of Eq. (36) by Eq. (37) gives

$$U = v_0 \log \frac{l}{\varepsilon} + F_1,$$ (38)

where F_1 is continuous and finite as $M \to O$. The logarithmic term in Eq. (38) gives the asymptotic expression of U as $M \to O$ "non-axially" (i.e. when $\psi_0 < \psi_1 < \pi - \psi_0$ for some small and finite ψ_0; cf. the discussion in Section 3).

Now, let $\mathscr{C}$ be an arc of a smooth curve of linear density v. Let O denote the origin at one end point of $\mathscr{C}$, Ox the x-axis along the tangent direction to $\mathscr{C}$ at O, Q the orthogonal projection of P to Ox, $\phi = \widehat{MOP}$ and $\psi = \widehat{MOQ}$ (see Fig. 4b). If s is the arc-length from O, P a generic point of $\mathscr{C}$ and $r = \overline{PM}$, then the Newtonian potential V associated with $\mathscr{C}$ and evaluated at an external point M is given by

$$V = \int_{\mathscr{C}} \frac{v\,\mathrm{d}s}{r}.$$ (39)

We want to determine the asymptotic expression of V as $M \to O$ "non-axially", i.e. when $\psi_0 < \psi_1 < \pi - \psi_0$. Let us sub-divide the arc C into two parts, $\mathscr{C}_1$, entirely contained in the neighbourhood of O, and the complement to $\mathscr{C}$, $\mathscr{C}^*$. From considerations of local geometry of $\mathscr{C}_1$ near O, and the
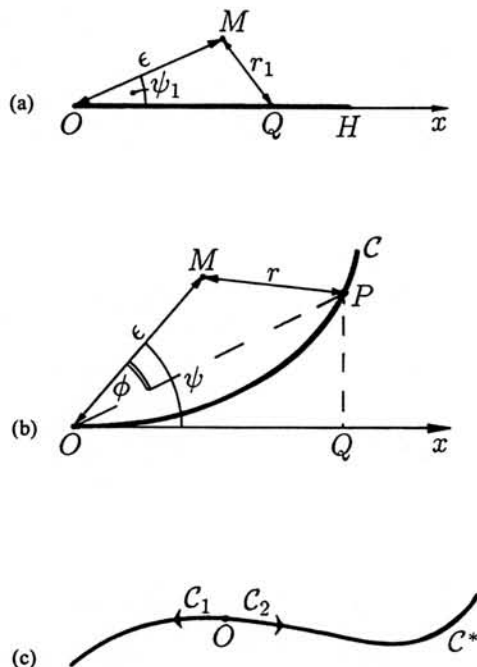
Fig. 4. (a) Potential effect of the segment OH at a neighbouring external point M. (b) Potential effect of the arc $\mathscr{C}$ at an external point M near $\mathscr{C}$. (c) $\mathscr{C}$ can be sub-divided into three parts: $\mathscr{C}_1$ and $\mathscr{C}_2$ adjacent and contiguous to O on either side (whose points are entirely contained in the neighbourhood of O) and the complement to $\mathscr{C}$, $\mathscr{C}^*$.

fact that the line potential associated with $\mathscr{C}^*$ remains finite as $M \to O$, we can show (LC, pp. 6–8) that asymptotically we have

$$V^{(a)} = v_0 \log \frac{l}{\varepsilon} \quad \text{(LC, p. 8, Eq. (13))}, \tag{40}$$

where $\varepsilon = \overline{OM}$, v_0 is the linear density at O, and l is a finite length ($l \ll R$, the radius of curvature of $\mathscr{C}$ at O).

The situation is easily generalized when O is a generic (internal) point of an arc $\mathscr{C}$. By considering only the contributions from neighbouring arcs $\mathscr{C}_1$ and $\mathscr{C}_2$ adjacent to O on either side (since the remaining part $\mathscr{C}^*$ gives finite contribution; see Fig. 4c), we have

$$V^{(a)} = 2v_0 \log \frac{l}{\varepsilon} \quad \text{(LC, p. 8, Eq. (14))}, \tag{41}$$

which is the asymptotic value at any point of $\mathscr{C}$.

Finally, consider a slender tube $\mathscr{T}$ of uniform (volume) density ρ and centreline $\mathscr{C}$. $\mathscr{T}$ is subdivided into a bundle of longitudinal fibres χ (Fig. 5) that run parallel along $\mathscr{C}$ without twist. Let σ be the area of the cross-section of $\mathscr{T}$ (not necessarily circular), $d\sigma_P$ the elementary area centred at P, and s the arc-length on $\mathscr{C}$. We assume that at a given point Q of σ passes only one fibre χ, which has the same arc-length parametrization and local geometry of $\mathscr{C}$ ($\mathscr{C} \equiv \chi$ when $Q \equiv P$). The
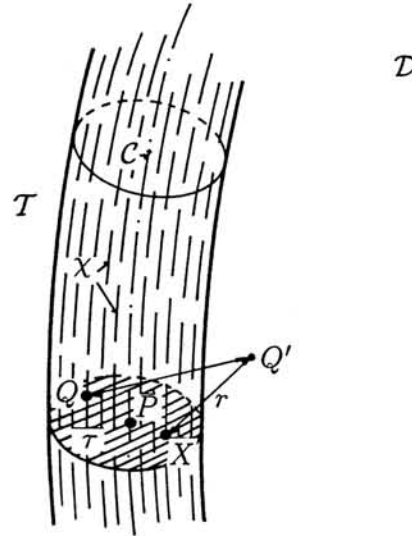
Fig. 5. Decomposition of $\mathcal{T}$ into a bundle of longitudinal fibres χ.

elementary volume at Q is therefore given by $d\mathcal{V} = D_Q\, d\sigma_Q\, ds$, where D_Q, continuous and finite, is the determinant at Q.

The Newtonian potential associated with $\mathcal{T}$ and evaluated at Q' is defined by

$$U_{Q'} = \int_{\mathcal{T}} \frac{\rho}{r}\, d\mathcal{V}, \tag{42}$$

where $r = \overline{XQ'}$ and X denotes a generic source point. Thus, Eq. (42) becomes

$$U_{Q'} = \int \left(\int_{\chi} \frac{D_Q\rho}{r}\, ds \right) d\sigma_Q = \int V_{Q'}\, d\sigma_Q, \tag{43}$$

where the integral given by $V_{Q'}$ is now interpreted as the line potential associated with the fibre χ of linear density $v = D_Q\rho$. Since $\max |Q - Q'| = \varepsilon \ll R$, where R is the radius of curvature of $\mathscr{C}$ at P, using Eq. (41) we can now evaluate the asymptotic form of $U_{Q'}$ given by

$$U_{Q'}^{(a)} = 2\int D_Q\rho_Q \log \frac{l}{\overline{QQ'}}\, d\sigma_Q \quad \text{(LC, p. 11, Eq. (3))}, \tag{44}$$

where D_Q and ρ_Q are the values at Q.

Now, since D_Q and its derivatives are continuous over σ, we can Taylor expand $D_Q\rho_Q$ near the mean value ρ_0, and write

$$D_Q\rho_Q = \rho_0 + (Q - P) \cdot F_1(Q,P) + (P - P_0) \cdot F_2(P,P_0), \tag{45}$$

where F_1 and F_2 are some finite functions on σ. The asymptotic expression for the Newtonian potential of $\mathscr{T}$ at Q' is therefore given by

$$U_{Q'}^{(a)} = 2\rho_0 \int \log \frac{l}{\overline{QQ'}} \, \mathrm{d}\sigma_Q \quad \text{(LC, p. 12, Eq. (8)).} \tag{46}$$

The asymptotic expression for the gradient of Eq. (46) is calculated using an earlier result of potential theory (Levi-Civita, 1908b), and we write

$$\nabla_{Q'}^{(a)} U_{Q'} = \nabla_{Q'} U_{Q'}^{(a)} + \tfrac{1}{2} U_{Q'}^{(a)} cn \quad \text{(LC, p. 18, Eq. (19)),} \tag{47}$$

where c and n are curvature and unit normal to $\mathscr{C}$ at P. It is convenient to write the gradient on the right-hand side of Eq. (47) as the sum of a longitudinal contribution (along $\mathscr{C}$) and a meridional contribution (in the normal plane to $\mathscr{C}$ at P). Thus

$$\nabla_{Q'}^{(a)} U_{Q'} = 2\rho_0 \overline{\nabla} \lambda_{Q'} + 2 \frac{\mathrm{d}}{\mathrm{d}s} (\rho_0 \lambda_{Q'}) t + \rho_0 \lambda_{Q'} cn \quad \text{(LC, p. 18, Eq. (19')),} \tag{48}$$

where $\lambda_{Q'}$ is the logarithmic potential

$$\lambda_{Q'} = \int \log \frac{l}{\overline{QQ'}} \, \mathrm{d}\sigma_Q, \tag{49}$$

and $\overline{\nabla} \lambda_{Q'}$ is the two-dimensional gradient referred to the meridional plane at P.

5.2. Asymptotic expressions for vector potential and curl

Consider now a vector field $\omega = \omega t_Q$ distributed on σ along the tangents t_Q to χ at Q ($t_Q \equiv t$ when $Q \equiv P$). The vector potential at Q' associated with ω is defined by

$$A_{Q'} = \int_{\mathscr{T}} \frac{\omega}{r} \, \mathrm{d}\mathscr{V}, \tag{50}$$

where $r = \overline{QQ'}$ and $A_{Q'} = (A_1, A_2, A_3)$ with respect to the Cartesian reference $Ox_1 x_2 x_3$. By substituting $p = \omega_Q t_{Qi}$, we can interpret each component of the vector potential as Newtonian potential and calculate the gradient. By replacing t_{Qi} ($i = 1, 2, 3$) with t_i (with an error of the order of ε) and using Eq. (48), we have

$$\nabla_{Q'}^{(a)} A_i = 2 \frac{p}{\sigma} t_i \overline{\nabla} \lambda_{Q'} + 2 \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{p}{\sigma} t_i \lambda_{Q'} \right) t + \frac{p}{\sigma} t_i \lambda_{Q'} cn \quad \text{(LC, p. 22, Eq. (4)),} \tag{51}$$

where p denotes the flux of ω through σ.

The asymptotic expression for the curl of the vector potential $A_{Q'}$ is calculated using Eq. (51). To do this we re-write the curl as

$$\nabla_{Q'} \times A = \sum_{i=1}^{3} \left(\frac{\partial A_i}{\partial x_{i+2}} \hat{e}_{i+1} - \frac{\partial A_i}{\partial x_{i+1}} \hat{e}_{i+2} \right)$$

$$= \sum_{i=1}^{3} \nabla_{Q'} A_i \times \hat{e}_i \quad \text{(LC, p. 23, Eq. (7')),} \tag{52}$$

where $\{\hat{e}_i\}$ ($i = 1, 2, 3$) is the basis of $Ox_1x_2x_3$ (with $x_1 \equiv x_4$ and $x_2 \equiv x_5$). By identifying the orthogonal basis with the Frenet triad $\{t, n, b\}$, we have

$$(\nabla_{Q'} \times A)^{(a)} \equiv \nabla_{Q'}^{(a)} \times A = 2\frac{p}{\sigma}\overline{\nabla}\lambda_{Q'} \times t + \frac{p}{\sigma}\lambda_{Q'}cb \quad \text{(LC, p. 23, Eq. (11))}, \tag{53}$$

which gives the asymptotic expression for the curl of the vector potential $A_{Q'}$.

5.3. Application of the asymptotic potential theory to vortex filament motion

Let us identify $\mathcal{T}$ with the vortex filament in $\mathcal{D}$ and apply the asymptotic results derived above to evaluate the induced velocity. By $\omega_Q = \nabla_Q \times v$ (note the factor $1/2$ in Eq. (1) of LC, p. 24), Eq. (50), and the usual conditions on v, we have $v_{Q'} = (1/4\pi)(\nabla_{Q'} \times A)$. The asymptotic expression for the induced velocity is derived using Eq. (53), and is given by

$$v_{Q'}^{(a)} = V_{Q'} + v_P, \tag{54}$$

where

$$V_{Q'} = 2\frac{\Gamma}{\sigma}\overline{\nabla}\lambda_{Q'} \times t \quad \text{(LC, p. 27, Eq. (9))}, \tag{55}$$

$$v_P = \Gamma k cb \quad \text{(LC, p. 26, Eq. (8))}, \tag{56}$$

with $\Gamma = p/4\pi$ =const, and

$$k = \frac{1}{\sigma^2}\int_\sigma \lambda_{Q'}\,\mathrm{d}\sigma_{Q'} \quad \text{(LC, p. 18, Eq. (21))}, \tag{57}$$

a configuration parameter function of s. The velocity is thus given by two contributions: the first, $V_{Q'}$, induces an area-preserving motion in the meridional plane that allows planar deformations (remember that $\overline{\nabla}$ is a two-dimensional operator; that σ is an invariant of the motion is a consequence of $\nabla \cdot V_{Q'} = 0$ in the meridional plane, which also implies, by Eq. (57), asymptotic invariance of the configuration parameter k; see Da Rios, 1910, pp. 360–365; 1911; 1916a). The second contribution, v_P, is proportional to the configuration parameter k, the local curvature $c = R^{-1}$, and induces a motion of the vortex along the binormal direction at P.

Let us consider the particular case of circular cross-section of diameter ε ($\varepsilon \ll R$) and area $\sigma = \pi\varepsilon^2/4$. If the centre is O, and $\varrho' = \overline{OQ'}$, the logarithmic potential $\lambda_{Q'}$ satisfies the Poisson equation on σ, i.e.

$$\Delta\lambda_{Q'} = \frac{1}{\varrho'}\frac{\mathrm{d}}{\mathrm{d}\varrho'}\left(\varrho'\frac{\mathrm{d}\lambda}{\mathrm{d}\varrho'}\right) = -2\pi, \tag{58}$$

with the condition that $\lambda_{Q'}$ remains finite at O. Integrating Eq. (58), we have

$$\lambda_{Q'} = \frac{\sigma}{2}\left(1 - \frac{4\varrho'^2}{\varepsilon^2}\right) + \sigma\log\frac{2l}{\varepsilon} \quad \text{(LC, p. 19, Eq. (23))}, \tag{59}$$

which reduces (dropping the suffix) to $\lambda = \sigma\log(2l/\varepsilon)$ for points on the vortex boundary. Therefore, by Eq. (59) and taking $\varepsilon = 2a$ (a radius of the cross-section), we have

$$\frac{\overline{\nabla}\lambda \times t}{\sigma} = \frac{1}{a}(\cos\theta\, b - \sin\theta\, n) = \frac{1}{a}\hat{q}, \tag{60}$$

where θ is a polar angle in the cross-sectional plane ($\theta = 0$ along n) and $\hat{q}$ is a rotational unit vector about $\mathscr{C}$. Hence, by Eq. (59), the asymptotic expression for the induced velocity at boundary points reduces to

$$v^{(a)} = \frac{p}{2\pi a}\hat{q} + \frac{p}{4\pi R}\log\frac{l}{a}b \quad \text{(LC, p. 25, Eq. (3))}. \tag{61}$$

It may be interesting to note that Eq. (61) was first derived by Da Rios in his study of fluid motion in the interior of vortex tubes with general cross-sections (see Da Rios, 1911, Eq. (II)) and independently re-derived by Batchelor in 1967 (cf. Section 7, p. 510 of his book).

By averaging over the cross-section, Eq. (54) reduces to Eq. (56). Thus, for a point $P = P(s,t)$, we have

$$v_P(s,t) = \frac{\partial P}{\partial t} = \Gamma kcb \quad \text{(LC, p. 30, Eq. (8'))}, \tag{62}$$

which preserves arc-length (and therefore total length of the vortex).

6. Stationary solutions of the intrinsic equations and stability of circular vortex filaments

Eq. (62), with $v_b = \Gamma kc$, provides a slight generalization of the localized induction Eq. (16). Combining this equation with the intrinsic Eqs. (32) and (33) (appropriately re-scaled with $t_1 = \Gamma t$; cf. LC, p. 38, Eqs. (I)), we can search for stationary solutions. By setting $\dot{c} = \dot{\tau} = 0$, we have

$$\left(\frac{B^2}{k^3 c^4} - \frac{(kc)''}{c}\right)' - c(kc)' = 0 \quad \text{(LC, p. 39, Eq. (3))}, \tag{63}$$

where $k = k(s)$ is now a known function of s and $B = k^2 c^2 \tau$ is an arbitrary constant of integration. Eq. (63) is an ordinary differential equation of the third order in the unknown $c(s)$ and can be regarded as the resolvent equation.

6.1. Rectilinear, circular and helical vortex filament solutions

Consider vortex filaments of uniform thickness. The configuration parameter k is taken constant with $h = B/k^2$. Eq. (63) reduces to

$$\left(\frac{h^2}{c^4} - \frac{c''}{c}\right)' - cc' = 0, \quad \tau = \frac{h}{c^2}. \tag{64}$$

The first of Eq. (64) is integrated by quadratures: taking $c^2 = -4\xi$, the equation becomes

$$\xi'^2 = -\tfrac{1}{4}h^2 - 2a_2\xi - 4a_1\xi^2 + 4\xi^3 \quad \text{(LC, p. 41, Eq. (7))}, \tag{65}$$

which is integrable by elliptic functions. In fact, by taking $c^2 = -4\wp - \tfrac{4}{3}a_1$, Eq. (65) reduces to the canonical Weierstrass equation (see, for example, Abramowitz and Stegun, 1972)

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3, \tag{66}$$

with invariants (in Levi-Civita's reference frame)

$$g_2 = -2a_2 - \tfrac{4}{3}a_1^2, \quad g_3 = -\tfrac{1}{4}h^2 - \tfrac{2}{3}a_1 a_2 - \tfrac{8}{27}a_1^3, \tag{67}$$

where a_1, a_2 and h are arbitrary constants of integrations. The Weierstrass elliptic function $\wp(s)$, solution to Eq. (66), is a second-order periodic function, completely determined by the invariants (67).

The simplest set of stationary solutions is given by the family of ∞^2 circular helices ($c = \text{const}$ and $\tau = \text{const}$) and ∞^1 circles. New helical solutions were derived by Da Rios (see p. 135, 1906, and his discussion of helical vortices, 1916c) and analysed further by Levi-Civita (1932a). Taking R as the radius of the circular cylinder on which the helix is inscribed and $\hat{k}$ the unit vector directed along the axis of helical symmetry, the tangent to the helix can be written as $t = \hat{k}\cos\theta + t_1\sin\theta$, where $t_1\sin\theta$ is the orthogonal projection of t onto the plane normal to $\hat{k}$. The binormal unit vector is thus given by

$$b = \cos\theta\,(\hat{k} \times n) \pm \sin\theta\,\hat{k} \quad \text{(LC, p. 48, Eq. (29))}, \tag{68}$$

where the positive or negative sign denotes right- or left-handed helices. Denoting by Q the projection of a point vortex P onto the central axis, we have $n = -(P - Q)/R$, and $c = (\sin^2\theta)/R$, so that by Eq. (62) the velocity of P is

$$v_P = \pm \frac{\Gamma k \sin^3\theta}{R}\,\hat{k} - \frac{\Gamma k \sin^2\theta \cos\theta}{R^2}\,\hat{k} \times (P - Q), \tag{69}$$

with a resultant helical motion.

6.2. Vortex solutions in the shape of planar loops and loop-generated configurations

New stationary solutions in the shape of planar loops and loop-generated configurations were discovered by Da Rios in various papers (1906, 1931a, b, c, 1933a, b; see also Levi-Civita, 1932a, pp. 41–47). These solutions are found setting $\tau = 0$ and integrating Eq. (64) directly by hyperbolic functions. As Da Rios showed (1931a; hereafter denoted by DR2), these solutions correspond to planar loops revolving without change of shape around a central axis. To see this, let xy denote the solution plane and x the rotation axis. According to LIA, we search for solutions for which

$$c(s) \equiv \frac{|\mathrm{d}\varphi|}{\mathrm{d}s} = Ky(s), \tag{70}$$

where φ is the angle between the curve tangent and the x-axis, K is a constant and $y(s)$ denotes distance of the solution curve from x. By definition of tangent, we have

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \cos\varphi, \quad \frac{\mathrm{d}y}{\mathrm{d}s} = \sin\varphi. \tag{71}$$

Eqs. (70) and (71) can be combined and integrated directly. Noting the symmetry with respect to the y-axis and choosing the positive sign for y, we have

$$x = \frac{1}{\sqrt{K}} \int \frac{1 - 2\sin^2(\varphi/2)}{(\sin^2(\varphi/2) + h)^{1/2}}\,\mathrm{d}(\varphi/2), \quad y = \frac{2}{\sqrt{K}}(\sin^2(\varphi/2) + h)^{1/2} \tag{72}$$

(DR2, pp. 723–724) (h constant of integration), explicitable in terms of elliptic functions.

Real values are given by $h > -1$. Three families of solutions were found by Da Rios and are reproduced in Fig. 6. The first family corresponds to $h = a^2 > 0$ and is represented by "pseudo-cycloids" (family I, Fig. 6a; note that at double points LIA is ill-defined). For $h = 0$ we have the second family of solutions represented by "pseudo-catenaries" (family II, Fig. 6b). In this case the first of Eqs. (72) can be integrated and becomes

$$x = \log \tan(\varphi/4) + 2\cos(\varphi/2) \quad \text{(DR2, p. 728, Eq. (12'))}, \tag{73}$$

with $x = 0$ at $y = \pi$. The corresponding intrinsic equation is given by

$$R(s) = \frac{1}{c(s)} = \frac{a}{4}(e^{s/a} + e^{-s/a}) \quad \text{(DR2, p. 728, Eq. (14))}, \tag{74}$$

and was derived by Da Rios in 1906 (cf. DR1, the last equation of p. 134, where by mistake $a/2$ is written instead of $a/4$). Alternatively, we can integrate the intrinsic equation (64) (with $h = 0$) and obtain

$$c(s) = \frac{2}{\cosh s} = 2\operatorname{sech} s \quad \text{(LC, p. 42, Eq. (13))}. \tag{75}$$

This solution was analysed by Levi-Civita in 1932 (cf. LC, pp. 41–47 where $a = 1$; see also his curve reproduced in Fig. 6d).

For $-1 < h < 0$ we have the third family of solutions given by the curves of Fig. 6c (family III), called "fluvial sinosoids" for their resemblance to the time-evolution of river-beds (another theme investigated by Da Rios). Their sinusoidal shape changes as h varies, with limit cases represented by the figure of horseshoe (not shown in Fig. 6c) and the figure of eight (as that one shown superimposed in Fig. 6c).

These three families of curves are actually well known in the scientific literature, since they represent equilibria of thin elastic rods (Bernoulli elasticae). As pointed out by Hasimoto (1971), there is indeed an analogy between vortex filaments and elastic rods (of which Da Rios and Levi-Civita seem to be unaware). By differentiating Eq. (70) with respect to s and using the second of Eqs. (71), we obtain the well-known equation for Bernoulli's elastica (Love, 1944, pp. 401–405), that is

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}s^2} = K\sin\varphi. \tag{76}$$

It is therefore not surprising that Euler's (1744) original sketches of his integral solutions to Eq. (76) (see, for example, Truesdell, 1983, p. 297) are the same curves reproduced in Fig. 6!

Under LIA, these loops revolve uniformly about the x-axis. This circumstance led Da Rios (1931c, 1933a) to envisage the existence of loop-generated vortex configurations called "rotating" vortices. These configurations were given by a family of n identical loops, each displaced from the previous by $\pi/2^{(n+1)}$ around a common rotational axis of revolution.

A combination of rotation and translation under LIA would equally give stationary solutions (since if $c = Ky$ is a solution, then $c = K_1 + Ky = Ky_1$ is also a solution). The "pseudo-catenary" solution is one remarkable example: it corresponds to the Hasimoto soliton solution (1972; cf. Eq. (3.8), p. 480) with $\tau = 0$. Moreover, the vortex surface that Da Rios envisaged as swept out by the loop during the motion is indeed a well-known soliton surface of minimum energy (Sym, 1984; Ricca,
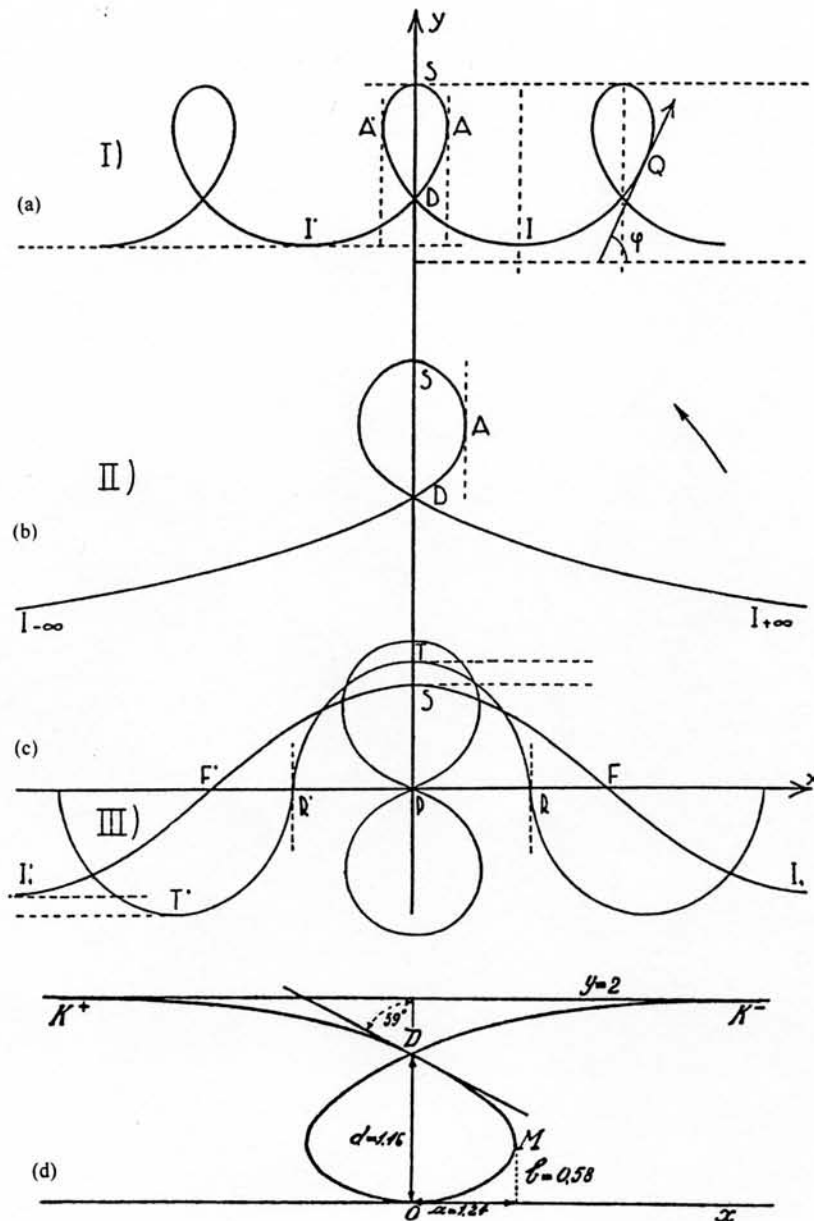
Fig. 6. Three families of vortex filament solutions in the shape of planar loops found by Da Rios in 1906 (DR1) and 1931 (DR2). The curves shown are reproduced from the original works of Da Rios and Levi-Civita. (a) "Pseudo-cycloid" (family I, $h > 0$; DR2); (b) "pseudo-catenary" (family II, $h = 0$; DR2); (c) "fluvial sinosoids" (family III, $-1 < h < 0$; DR2); (d) the "pseudo-catenary" solution (family II, $h = 0$, $a = 1$; LC) was also analysed by Levi-Civita. All these curves correspond to the well-known Euler's (1744) solutions to Bernoulli's elastica problem (cf. Truesdell, 1983, p. 297).
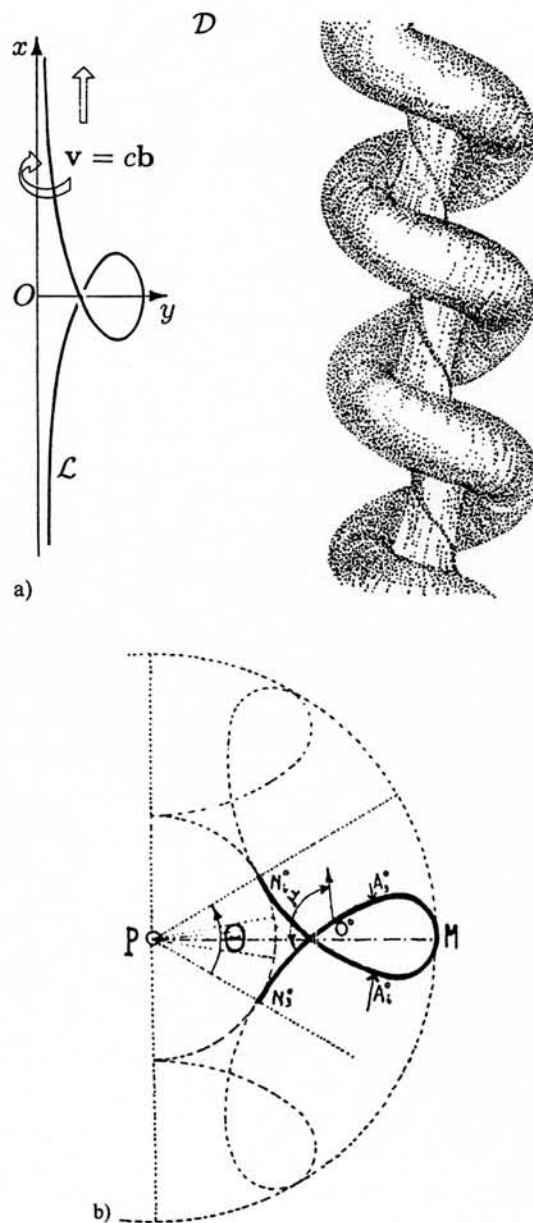
Fig. 7. (a) The loop generated surface (on the right) envisaged by Da Rios as swept out by the helical motion of his "pseudo-catenary" solution (on the left) is now known to be a soliton surface of minimum energy (Sym, 1984; Ricca, 1992). This is actually the surface swept out by the Hasimoto (1972) soliton, with $v = cb$ and $\tau = \text{const} \neq 0$. (b) The solution shown was studied by Da Rios (1933b) and is a particular case of the Kida (1981) class of steady solutions.

1991b, 1992) (Fig. 7a). A second example is given by the "pseudo-helical" solution reproduced in Fig. 7b, which was obtained directly from the intrinsic study of Eqs. (64) and (66) (see Da Rios, 1933b). This last solution is a particular case of the Kida (1981) class of steady solutions discovered almost 50 years later.

6.3. Stability of circular vortex filaments

Let $c_0(s)$ and $\tau_0(s)$ be stationary solutions of the intrinsic equations (64) ($k = $ const), and consider small amplitude perturbations from the steady state, so that curvature and torsion of the perturbed curve are given by

$$c(s,t) = c_0(s) + \varepsilon(s,t), \quad \tau(s,t) = \tau_0(s) + \eta(s,t). \tag{77}$$

By substituting Eqs. (77) into Eqs. (34) and (35) and taking terms up to the first order, we have (Eqs. (32) and (33) of LC, p. 50)

$$
\begin{aligned}
\dot{\varepsilon} &= f = (c_0\eta + \tau_0\varepsilon)' + c_0'\eta + \tau_0\varepsilon', \\
\dot{\eta} &= g = \left[2\tau_0\eta - (\varepsilon''/c_0) + (c_0''\varepsilon/c_0^2)\right]' - (c_0\varepsilon)',
\end{aligned} \tag{78}
$$

now linear in ε, η and their derivatives with respect to s.

These equations can be used to study the stability of circular vortex filaments. In the case of a circle, we have $c_0 = $ const and $\tau_0 = 0$. With these assumptions (and appropriate re-scaling so that $c_0 = 1$), Eqs. (78) reduce to

$$\dot{\varepsilon} = \eta', \quad \dot{\eta} = -\varepsilon''' - \varepsilon', \tag{79}$$

and by differentiating the first equation with respect to time and the second equation with respect to arc-length and eliminating $\dot{\eta}'$, we have

$$\ddot{\varepsilon} + \varepsilon'''' + \varepsilon'' = 0 \quad \text{(LC, p. 51, Eq. (34)),} \tag{80}$$

with initial conditions given by $\varepsilon_0(s)$ and $\dot{\varepsilon}_0(s)$, both uniform periodic functions of period 2π. Since Eq. (80) is linear, the initial conditions will be of the kind $A\exp(ins)$, $B\exp(ins)$ (i imaginary unit), with A and B arbitrary constants and $n \neq \pm 1$ integer. Note that the condition $n \neq \pm 1$ follows from the requirement of closure for the perturbed curve: if the curve is closed, then

$$\int_0^{2\pi} t(s)\, ds = 0, \tag{81}$$

which gives

$$\int_0^{2\pi} \varepsilon_0(s) e^{\pm is}\, ds = 0 \quad \text{(LC, p. 52, Eq. (37)).} \tag{82}$$

This means that in the Fourier representation of the periodic function $\varepsilon_0(s)$, terms in $\exp(\pm is)$ are missing. Hence, by taking $\varepsilon(s,t) = [\exp(ins)]q(t)$ with $n \neq \pm 1$, Eq. (80) reduces to

$$\ddot{q} + (n^4 - n^2)q = 0, \tag{83}$$

which is a second-order ODE with constant coefficients and initial conditions $q(0) = A$ and $\dot{q}(0) = B$. Existence and uniqueness of the solution was studied by Levi-Civita (1932b), who showed that the integral solution is bounded and periodic in time for $|n| > 1$. This means that circular vortex filaments are asymptotically stable to linear perturbations, a result successively re-discovered, for example,

by Kambe and Takao (1971). From Eq. (83), we deduce the periods of vibration. In appropriate dimensional form, these are given by

$$T_n = \frac{2\pi R^2}{\Gamma k (n^4 - n^2)^{1/2}} \quad (n = 2, 3, \ldots) \quad \text{(LC, p. 54, Eq. (40))}, \tag{84}$$

with a fundamental period given by $n = 2$. From inspection of the form of the solution curve, the general integral can be seen as the envelope of a family of simple progressive and regressive waves moving in opposite directions along the vortex filament, whose speed (c_n) and wavelength (λ_n) are given by

$$c_n = \frac{\Gamma k}{R} n (1 - 1/n^2)^{1/2}, \qquad \lambda_n = \frac{2\pi R}{n} \quad (n = 2, 3, \ldots). \tag{85}$$

7. Conclusions

The work of Da Rios and Levi-Civita on vortex filament motion and asymptotic potential theory spanned a period of almost 30 years, from 1906 to 1933, and represents one of the first major contributions to three-dimensional vortex filament dynamics. Despite many publications, their work remained almost unnoticed until recently (Germano, 1983; Ricca, 1991a). With this paper we have reviewed and discussed for the first time their contribution, including some material found very recently and published in the years 1916–1933. The results presented include the conception of the localized induction approximation (LIA) for the induced velocity of thin vortex filaments, the derivation of the intrinsic equations of motion, the asymptotic potential theory applied to vortex filament motion, the derivation of stationary solutions in the shape of helical vortices and loop-generated vortex configurations, and the stability analysis of circular vortex filaments.

In the light of modern developments of non-linear dynamics and vorticity, their work strikes for modernity and depth of results. Even more striking is the fact that this work remained obscure for almost a century. The partial lack of interest of the scientific community of the time and the lack of acknowledgements in contemporary publications (cf. Villat's book, 1930) did not give much chance to post-war authors (see, for example, Truesdell, 1954) to get to know these results.

After the re-derivation of LIA by Arms and Hama (1965) and the independent re-discovery of the intrinsic equations by Betchov (1965), most of the work of Da Rios has been gradually re-discovered. For example, the steady solutions and loop-generated vortex configurations have been re-derived and interpreted in terms of soliton dynamics (Hasimoto, 1972; Sym, 1984) and travelling waves (Kida, 1981). It is interesting to note that even after a full generalization of the intrinsic equations to higher dimensional manifolds (Ricca, 1991b), Da Rios's equations are still being re-derived in one form or another (see, for example, Langer and Perline, 1991; Nakayama et al., 1992).

If LIA and Da Rios's intrinsic equations are now very much at the core of integrable one-dimensional systems and modern soliton theory (Langer and Perline, 1991; Nakayama et al., 1992; Ricca 1993, 1995), the work of Levi-Civita is fundamental for the mathematical formulation of potential theory and capacity theory for slender tubes (Landkof, 1972).

Acknowledgements

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