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Geometric and Topological Aspects of Vortex Filament Dynamics under LIA

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Abstract: Geometric and topological aspects associated with integrability of vortex filament motion in the Localized Induction Approximation (LIA) context (which includes a family of local dynamical laws) are discussed. We show how to interpret integrability in relation to the Biot-Savart law and how soliton invariants can be interpreted in terms of global geometric functionals of knotted solutions. Under the basic (zeroth-order) LIA, we prove that vortex filaments in the shape of torus knots $\mathcal{T}_{p,q}$ (p, q co-prime) with $(q/p) > 1$ are stable, whereas those with $(q/p) < 1$ are unstable.

1 Localized Induction Approximation (LIA) and integrability

The velocity $\boldsymbol{u}(\boldsymbol{X})$ ($\boldsymbol{X}$ position vector in $\mathbb{R}^3$) with which an isolated thin vortex filament propagates in an incompressible perfect fluid is given by the Bio-Savart integral, which is a global integral functional of the vorticity distibution $\boldsymbol{\omega}$. With reference to the vortex filament axis (parametrised by the arc-length s), $\boldsymbol{u}$ can be decomposed in intrinsic components (u_t, u_n, u_b) along the unit tangent vector $\boldsymbol{t}$ $(= \boldsymbol{X}' = \partial \boldsymbol{X}/\partial s)$, unit principal normal $\boldsymbol{n}$ and unit binormal $\boldsymbol{b}$ to the vortex axis.

Given the thinness of the vortex and neglegting long-distance effects and self-interactions, we can develop an asymptotic theory and approximate the motion of the vortex with simple local laws [1]. We call this approach 'Localized Induction Approximation' (LIA for short). Under LIA we have a family (or, better, a hierarchy) of local laws of increasing complexity, depending on the degree of sophistication of the physical model and the amount of information that we want to take into account. The simplest example, which may be considered as a zeroth-order LIA, is given by a very crude approximation to Biot-Savart that after appropriate rescaling can be written as

$$\boldsymbol{u} = \dot{\boldsymbol{X}} = \boldsymbol{X}' \times \boldsymbol{X}'' = c\boldsymbol{b} \,, \tag{1}$$

with $u_t = u_n = 0$, $u_b = c$ (c curvature of the vortex axis), being everything a (sufficiently) smooth function of arc-length s and time t (over-dots denoting $\partial/\partial t$). Under the Hasimoto map [2] (which is a Madelung transformation of curvature c and torsion τ of the vortex axis to the complex plane) $\psi(s,t) = c(s,t)\exp[\int^s \tau(\xi,t)\,d\xi]$, $\psi \in \mathbb{C}$, eq. (1) can then be reduced to the (focusing) non-linear Schrödinger equation $-i\dot{\psi} = \psi'' + (1/2)|\psi|^2\psi$ (NLSE), which in one dimension is completely integrable, has soliton solutions and an infinite sequence of conserved quantities in involution [3,4]. A first refinement of the physical model is given by taking into account the presence of axial flow (measured by $F = $ cst.) in the vortex, so that the induced local velocity becomes [5]

$$\boldsymbol{u} = c\boldsymbol{b} + F\left(\frac{c^2}{2}\boldsymbol{t} + c'\boldsymbol{n} + c\tau\boldsymbol{b}\right) , \tag{2}$$

which is equivalent (by the Hasimoto map) to the focusing version of the modified Korteweg-de Vries equation (mKdV), the second Hamiltonian system in the NLSE hierarchy of integrable soliton equations (Hirota class). Note that in the absence of axial flow ($F = 0$) we recover eq. (1). Similarly, it is possible to take account of additional physical aspects, such as linear inhomogeneities and second order effects (due, for example, to particular local vorticity distribution), which will give more elaborate expressions for $\boldsymbol{u}$ while preserving integrability.

In general, it is interesting to inquire under which physical conditions integrability is preserved and to what extent physical properties are related to the underlying mathematical structure. A partial answer to the first point is based on the study of the Hamiltonian structure associated with soliton equations and their Poisson geometry. It is possible to show [6] that there is a recursive operator that generates the dynamical laws (such as those given by eqs. 1 and 2) associated with integrability. The construction can be explicited in the recursive formula

$$\begin{aligned} \dot{\boldsymbol{X}}^{(0)} &= c\boldsymbol{b} , \\ \dot{\boldsymbol{X}}^{(j+1)} &= \dot{\boldsymbol{t}}^{(j)} \times \boldsymbol{t} + \left[\int^s c(\xi,t)\left(u_b'^{(j)} + \tau u_n^{(j)}\right) d\xi\right] \boldsymbol{t} , \end{aligned} \tag{3}$$

where $j = 0, 1, 2, \ldots$. Here, two remarks can be made. First, it should be noted that the integral in the r.h.s. of (3) is an operator that preserve arc-length (these dynamical laws act as Killing fields on the vortex centreline, preserving total length, and therefore enstrophy). This means that in this context inextensibility of vortex filaments is a constitutive property of the integrable dynamical laws generated by (3). As a consequence, the dynamics of vortex filaments under stretching cannot be reconstructed by the recursive formula above. But this doesn't mean that an (intrinsic) stretching cannot generate integrable dynamics. An example is given by the velocity $\boldsymbol{u} = G\boldsymbol{t} + Hc\boldsymbol{b}$ (with G and H both constants), which generates the sine-Gordon equation (sG) [7], a completely integrable soliton equation not captured by the recursive formula (3).

A second point of interest about (3) concerns the term $\dot{\boldsymbol{t}} \times \boldsymbol{t}$ in the recursive construction of the $(j+1)$-th dynamical law. Since $\dot{\boldsymbol{t}} = \partial\dot{\boldsymbol{X}}/\partial s$, we can write

$$\dot{X}^{(j+1)} = \sum_{\nu=1}^{j+1} \mathcal{F}\left(\frac{\partial^\nu \dot{X}^{(0)}}{\partial s^\nu}\right) = \mathcal{R}\left(\dot{X}^{(0)}\right) , \tag{4}$$

where $\mathcal{F}$ and $\mathcal{R}$ are nonlinear operators acting on the zeroth-order LIA through all its derivatives (up to order $j+1$). Here $\mathcal{R}$ can be seen as an operator that acts similarly as in the construction of a Taylor series, so that it would seem appropriate to interpret the $(j+1)$th-LIA as an expansion of the Biot-Savart law around the original zeroth-order LIA (eq. 1).

2 Polynomial invariants and global geometric quantities

Under LIA integrability is preserved together with an infinity of conserved quantities in involution. As we add more physical information and move gradually from a local to a global approach (where the extreme is represented by Biot-Savart) most of these quantities tend to decay and we are left just with the most robust invariants (we know, for example, that kinetic energy, helicity, linear and angular momenta are conserved throughout this process [4]). A clear understanding of this important mechanism is still missing.

A geometric interpretation of the invariants associated with integrability is based on the application of the inverse of the Hasimoto map. Since the inverse is well defined on $\mathbb{C}$ we can derive a complete set of conservation laws (soliton invariants) expressed in terms of global geometric functionals $\mathcal{I}_n$ $(n = 1, 2, \ldots)$, i.e.

$$\mathcal{I}_n = \oint D_n(s,t)\, ds = \oint F[\psi(s,t)]\, ds = \text{cst.} , \tag{5}$$

where the densities $D_n(s,t)$ are function of curvature, torsion and their spatial derivatives. This geometric interpretation was totally unexpected and led us to a number of very interesting observations and new relationships between global geometry and topology [8] (in this regard we should enphasize that under LIA evolution the topology is conserved up to filament crossings, when the validity of LIA breaks down). In the case of eq. 1, for example, together with conservation of total length and total torsion, we have [4]

$$D_1 = c^2 , \quad D_2 = c^2\tau , \quad D_3 = c^2\tau^2 + c'^2 - (c^4/4) , \quad \ldots , \tag{6}$$

where, for example, D_1 can be interpreted as kinetic energy density and D_2 as helicity density $\boldsymbol{u} \cdot \nabla \times \boldsymbol{u}$ of the vortex filament. It is interesting to note that under (1) helicity can be written as [9]

$$\mathcal{H} = \kappa^2 Lk = \kappa^2 \left[Wr + \frac{1}{2\pi} \oint \tau\, ds \right] \tag{7}$$

where Lk denotes the linking number of the vortex (a topological invariant), Wr the writhing number (a geometric measure of the avarage number of apparent signed crossings of the filament axis) and κ the vortex circulation (constant).

Since in this context both helicity and total torsion are constant, the writhe is constant too (up to filament crossings). In actual fact it is possible to prove that

$$Wr \sim \oint (c^2 - 1)\tau \, ds = \text{cst.} \tag{8}$$

(a double check can be made by using (1) and the intrinsic eqs. for time derivatives of curvature and torsion).

On dimensional grounds we should note that since both curvature and torsion scale like the inverse of a length $[L^{-1}]$, the global geometric functionals $\mathcal{I}_n$ scale like $[L^{-n}]$. If this can be associated with the physical interpretation of a cascade process we cannot say, but it certainly allows us to construct a series of polynomial invariants (in a dummy variable) whose coefficients are the functionals $\mathcal{I}_n$. For a given knotted configuration and under specific LIA evolution, the series would then encapsulate both integrability and global geometry, but it wouldn't be a measure of topological invariance.

3 Kelvin's conjecture and stable torus knot solutions

Kelvin was the first to address the problem of steadiness and stability of knotted vortex filaments and in particular he conjectured (see [10], p. 123, ¶16) that thin vortex filaments in the shape of torus knots should be stable. This conjecture was never proved. We give here a demonstration based on eq. (1) and our previous work [4,8]. Wave solutions to (1) expressed in cylindrical polar coordinates (r, α, z) (smooth functions of s and t) and in the shape of torus knots $\mathcal{T}_{p,q}$ can be found by linear perturbation from the circular solution. Here p is the number of wraps around the torus in the longitudinal direction and q is the number of wraps in the meridian direction; p and q are positive co-prime numbers and $w = q/p$ is the winding number of the knot. Rewrite eq. (1) in cylindrical polar coordinates (cf. [8]), and let

$$r = r_0 + \epsilon r_1 \ , \quad \alpha = \frac{s}{r_0} + \epsilon \alpha_1 \ , \quad z = \frac{\hat{t}}{r_0} + \epsilon z_1 \ , \tag{9}$$

where r_0 is the radius of the unperturbed circular vortex filament, $\hat{t}$ is a scaled 'time' parameter and $\epsilon = o(1)$. The linearised set of equations is thus given by

$$\left. \begin{aligned} \dot{r}_1 = z_1'' \ , \quad \dot{\alpha}_1 = -\frac{1}{r_0^2} z_1' \ , \\ \dot{z}_1 = -r_1'' + \frac{2}{r_0^2} r_1 + 3\alpha_1' \ . \end{aligned} \right\} \tag{10}$$

Wave solutions in the shape of closed torus knots are obtained taking

$$r_1(s,t) = R(t) \cos \frac{ws}{r_0} \ , \ \alpha_1(s,t) = A(t) \sin \frac{ws}{r_0} \ , \ z_1(s,t) = Z(t) \cos \frac{ws}{r_0} \ , \tag{11}$$

and by (11) eqs. (10) are reduced to a linear system of first order differential equations, i.e.

$$\left.\begin{aligned} \dot{R} &= -\left(\frac{w}{r_0}\right)^2 Z\,, \quad \dot{A} = \left(\frac{w}{r_0}\right)\frac{1}{r_0^2}Z\,, \\ \dot{Z} &= \left[\frac{2}{r_0^2} + \left(\frac{w}{r_0}\right)^2\right] R + \left(\frac{3w}{r_0}\right) A\,, \end{aligned}\right\} \tag{12}$$

which can be solved by standard methods. With $(R, A, Z) \propto \exp(\rho t)$ (where $\rho \in \mathbb{C}$), eq. (12) is further reduced to the matrix system

$$\begin{pmatrix} \rho & 0 & \left(\frac{w}{r_0}\right)^2 \\ 0 & \rho & -\frac{w}{r_0^3} \\ \frac{2}{r_0^2} + \left(\frac{w}{r_0}\right)^2 & \frac{3w}{r_0} & -\rho \end{pmatrix} \begin{pmatrix} R \\ A \\ Z \end{pmatrix} = 0\,. \tag{13}$$

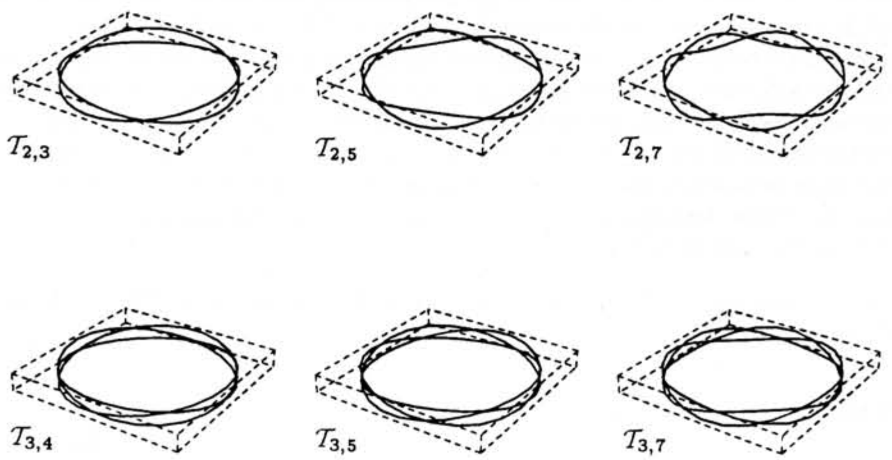

Fig. 1. Examples of stable vortex filaments in the shape of torus knots $\mathcal{T}_{p,q}$ with $q/p > 1$.

The characteristic equation is

$$\rho^3 + \left(\frac{w}{r_0}\right)^2 \left[\left(\frac{w}{r_0}\right)^2 - \frac{1}{r_0^2}\right] \rho = 0\,, \tag{14}$$

which is solved by $\rho = 0$ and (replacing $w = q/p$) by

$$\rho^2 = -\frac{1}{r_0^4}\left(\frac{q}{p}\right)^2 \left[\left(\frac{q}{p}\right)^2 - 1\right]\,. \tag{15}$$

If $(q/p) > 1$, then $\rho^2 < 0$ and we have stable wave solutions in the shape of torus knots. If $(q/p) < 1$, then $\rho^2 > 0$ and we have an exponential growth of the perturbations (instability). This means that in the class of torus knots $\mathcal{T}_{p,q}$, knotted vortex filaments with $(q/p) > 1$ are stable, whereas those with $(q/p) < 1$ are unstable: for example, a vortex filament in the shape of a standard trefoil knot $K_{2,3}$ is stable, but the knot $K_{3,2}$ (which is topologically equivalent to $K_{2,3}$) is unstable. This result is new and unexpected.

The analytical solutions expressed in terms of traveling-waves are given by

$$\left.\begin{aligned} r &= r_0 + \epsilon k_r \sin\left[\left(\frac{q}{p}\right)\frac{\eta}{r_0} + \beta_k\right] , \\ \alpha &= \frac{s}{r_0} + \epsilon\left(\frac{q}{p}\right)\frac{k_r}{r_0}\cos\left[\left(\frac{q}{p}\right)\frac{\eta}{r_0} + \beta_k\right] , \\ z &= \frac{\hat{t}}{r_0} + \epsilon k_r\left[1 - \left(\frac{p}{q}\right)^2\right]^{\frac{1}{2}}\cos\left[\left(\frac{q}{p}\right)\frac{\eta}{r_0} + \beta_k\right] , \end{aligned}\right\} \tag{16}$$

with $\eta = s - at$, where $a =$ cst. is the propagation velocity of the wave and k_r and β_k are constants. Details on the derivation of (16) can be found in [8], where questions of stability were not addressed. Equations (16) are in perfect agreement with the solutions found by Kida [11] (for the fully non-linear problem), who expressed them in implicit form, without addressing the problem of stability. With our method we have torus knot solutions in explicit analytical form as small amplitude perturbations from the circular solution, together with conditions for stability. These techniques are quite general and can be applied similarly to higher-order LIA dynamics.

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