Intrinsic equations for the kinematics of a classical vortex string in higher dimensions

Renzo L. Ricca

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge, England CB3 9EW

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We present here the natural algebraic-geometrical generalization of the Da Rios–Betchov intrinsic equations governing curvature and torsion of an isolated vortex string moving in an unbounded, perfect fluid flow. The filament is embedded in a manifold that can be assumed to be homeomorphic to an odd-dimensional Euclidean space, and whose connection we do not assume to be torsion-free. We suggest how to account for fluid compressibility in the ambient space by its geometrization, and we discuss some special cases of physical interest such as the torsion-free affine connection case and the Riemannian connection case. Finally, we point out the role our results might have in the context of soliton studies.

I. INTRODUCTION

In recent works,[1,2] the intrinsic equations of motion for a curved vortex line in the three-dimensional Euclidean space have been explicitly deduced in terms of the time evolution of its curvature and torsion. These results extended the particular set of Da Rios equations concerning the motion of an isolated vortex filament of negligible core size moving in a perfect fluid flow under the so-called local induction approximation[3] (LIA). The same equations were discovered independently by Betchov 60 years later.[4] We present here their natural algebraic-geometrical generalization, developing the calculation of the intrinsic equations of motion for the extensible vortex string, embedded instantaneously in a manifold locally homeomorphic to an odd-dimensional Euclidean space, whose connection is not assumed to be torsion-free.

The string is a one-dimensional classical field singularity, and our attempt here is to study its intrinsic kinematical behavior; though this kind of approach is not new and dates back to the early studies of hyperspaces and hypergeometry,[5] there are several reasons to believe that it deserves our attention, primary among these being the recent results in the field of soliton studies and nonlinear dynamics. The analysis of the motion of a one-dimensional extended object thought of either as a curved line or as the reference centerline of a real filament is intriguing, not only because of the specific intrinsic approach that we take up here, but also for the properties related to the surface the string describes during its motion. Our interest is partially motivated by the wide variety of physical examples having the classical string model as a natural reference: one-dimensional classical continuum spin systems, current density lines, vortex lines in perfect fluids, magnetofluid flows, superfluid He II, and type II superconductors.[6,7] This is owing principally to the deep formal analogies between the several fundamental field equations involved. Without loss of generality, we concentrate on the case of the vortex string in an external background ideal fluid flow field. In such a case the vorticity will be given by a pointwise distributed δ-Dirac singularity field along the line, with the vorticity vector and tangent vector everywhere parallel. The dynamic ingredient for the motion is due to the self-induced velocity generated by the potential action of the vorticity distribution;[8] for our purposes it is expressed through its intrinsic components, that is with reference to an orthonormal basis of the Frenet-Serret type on the hyperspatial curve. We should note that the self-energy of the vortex is logarithmically divergent because of the Green function connected to the Poisson equation. This is due to the relative distance between the observation point (where the field is measured) and the source point (where the vorticity unit source is placed), which tends asymptotically to zero. We shall deal, however, with a general formulation of the law of motion, considering all the intrinsic components to be contributive.

This work suggests how to take into consideration the (possible) compressibility of the ambient space in which the filament is thought to be initially embedded; this is significant, because the compressibility condition implies serious difficulties in managing the group of volume-preserving diffeomorphisms of the domain of fluid flow. An elementary example of such a situation is given by a gas in a three-dimensional Euclidean space whose density is a function of the spatial position of every material point; differences in the density distribution of the background flow field are, in fact, related to the divergence condition for the velocity field. The difficulty implicit in the divergence condition can be met by geometrizing the compressibility of the ambient space. This we can do in two steps: first we calculate the Jacobian by a pointwise mapping of the local density of the domain space, thus obtaining a uniform value; then we use this function to obtain the curvature tensor of the newly transformed manifold, which, beyond being simply a function of position, will be also a function of the Jacobian. Such a tensor could properly be defined a *compressor.*

Instead of working with compressible fluid in a Euclidean flat space, we will thus work in a general differentiable Riemannian manifold with fluid under the condition of zero divergence. In this way we interpret the local curvature of the manifold as the expression of the local density of the ambient space (Fig. 1).

The work is then organized as follows: in Sec. II we recall some fundamental definitions from classical

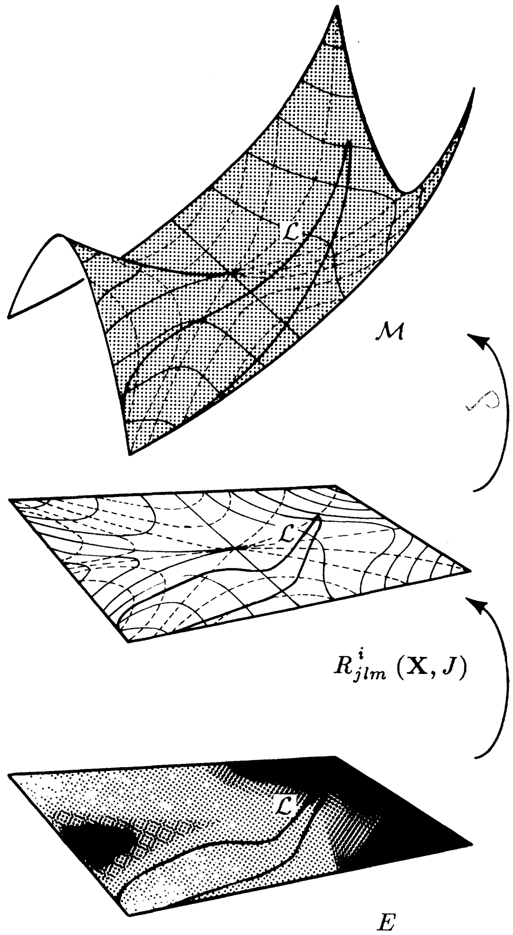

FIG. 1. Geometrization of the compressibility of the ambient (Euclidean) space by the particular tensor of curvature (here called *compressor*), a function of the Jacobian.

differential geometry, and we explain step by step the logic of the mathematical operations for deriving the intrinsic equations of motion. By applying this scheme in Sec. III, we derive the equations in their general form. In Sec. IV we deal with some cases of particular interest, pointing out the links between these equations and those of nonlinear dynamics. And we conclude in Sec. V with some remarks about the kinematics of curves and soliton studies.

II. GEOMETRICAL-KINEMATICAL PRELIMINARIES

Let $\mathcal{M}$ be a differential paracompact connected Hausdorff manifold of class C^∞ on $\mathbb{R}^{2n+1}$. $\mathcal{M}$ can be assumed to be locally homeomorphic to the odd-dimensional Euclidean flat space E and is without boundary. The position of any point $P\in\mathcal{M}$ will be denoted by

$$P=P(X^i),\quad i=1,\ldots,2n+1$$

where $\{X^i\}$ are general extrinsic coordinate components. Let us introduce the Jacobian $J=J(\zeta)$ according to the mapping

$$\zeta:\ \rho_0\to\rho,\quad 0<\zeta<\infty$$

where the domain ρ_0 is the uniform density on $\mathcal{M}$, and the image $\rho=\rho(\mathbf{X})$ is the density of the ambient space E, which is possibly filled by compressible fluid. The incompressibility on the manifold $\mathcal{M}$ is then assured by the existence of the inverse mapping ζ^{-1}, that is, by asking for strictly positive values for ζ. Notice that $\zeta=1$ separates expansion from compression.

From classical differential geometry we also assume the following general statements.[9,10] Let the *metric tensor* g be a smooth and positive-definite section of the bundle of the symmetric bilinear two-forms on $\mathcal{M}$. Given a C^r linear connection ∇ on the manifold $\mathcal{M}$, one can define a C^{r-1} tensor field T of type (1,2) by the relation

$$\mathsf{T}(\mathbf{Y},\mathbf{Z})=\tfrac{1}{2}(\nabla_{\mathbf{Y}}\mathbf{Z}-\nabla_{\mathbf{Z}}\mathbf{Y}-[\mathbf{Y},\mathbf{Z}])$$

and define a C^{r-1} tensor field R of type (3,1) by the relation

$$\mathsf{R}(\mathbf{Y},\mathbf{Z})(\mathbf{V},\mathbf{W})=[\nabla_{\mathbf{Y}}\nabla_{\mathbf{Z}}\mathsf{g}(\mathbf{W})-\nabla_{\mathbf{Z}}\nabla_{\mathbf{Y}}\mathsf{g}(\mathbf{W})-\nabla_{[\mathbf{Y},\mathbf{Z}]}\mathsf{g}(\mathbf{W})](\mathbf{V})\ ,$$

where $[\mathbf{Y},\mathbf{Z}]$ is the Lie derivative of $\mathbf{Z}$ with respect to $\mathbf{Y}$ and $\mathbf{Y},\mathbf{Z},\mathbf{V},\mathbf{W}$ are arbitrary C^{r+1} vector fields. We call T the *torsion tensor* and R the *Riemann (curvature) tensor*. Let us assume a *coordinate basis* $\{\partial/\partial X^i\}$. Then the tensor g at a point $P\in\mathcal{M}$ will be a symmetric tensor of type (0,2) at P given by

$$\mathsf{g}=g_{ij}dX^i\otimes dX^j\ .$$

In order to take into account the mapped incompressibility condition on the manifold $\mathcal{M}$ by the Jacobian J, we want to stress the dependence of the previous quantities on $\mathbf{X}$ and J. We assume for any infinitesimal coordinate displacement measured by the arc length dl on $\mathcal{M}$ the following relation:

$$dl^2=g_{ij}(\mathbf{X},J)dX^idX^j$$

and then

$$L^i_{jk}(\mathbf{X},J),\quad T^i_{jk}(\mathbf{X},J),\quad R^i_{jlm}(\mathbf{X},J)\ ,$$

where we denote by L^i_{jk} the affine connection coefficients, by $T^i_{jk}=\frac{1}{2}(L^i_{jk}-L^i_{kj})$, the components of the torsion tensor of the connection, and by R^i_{jlm}, the components of the curvature tensor of the connection. Given its particular function, the curvature tensor could be called a "compressor" (Fig. 1). We have indicated the affine connection by

$$L^i_{jk}=\Gamma^i_{jk}+\Upsilon^i_{jk}\ ,$$

where $-\Upsilon^i_{jk}$ is the *contorsion* and states the difference between the Riemannian connection given by Christoffel's symbols Γ^i_{jk} and the affine connection L^i_{jk} on $\mathcal{M}$.

Let us assume a not-necessarily symmetric affine connection on $\mathcal{M}$ and consider the hypercurve $\mathcal{L}_\tau$ as an instantaneous one-dimensional field embedded in $\mathcal{M}$.

Strictly speaking, $\mathcal{L}_\tau$ is the image of a mapping according to the following.[11–13]

Definition. We identify $\mathcal{L}_\tau \subset \mathcal{M}$ with the C^p bijection ($p > 2n+1$)

$$\Lambda:\ T \to \mathcal{M} \tag{2.1}$$

for a compact $T=[a,b]\subseteq \mathbb{R}^1$, such that $\Lambda(a)=\Lambda(b)$ $\forall a,b \in \mathbb{R}^1$, which associates the point

$$\mathcal{L}_\tau(s)=[X^1(s,\tau),\ldots,X^{2n+1}(s,\tau)],\quad \forall \tau \in I \subseteq \mathbb{R}_+$$

with $s \in T$. Then $\mathcal{L}$ is the geometrical configuration associated with the image of the mapping (2.1) and is parametrized by the

$$X^i = X^i[s(s_0,\tau),\tau]\ , \tag{2.2}$$

where

$$s(s_0,\tau_0)=s_0,\quad s(0,\tau)=0\ . \tag{2.3}$$

We have denoted by $s \in T$ the curvilinear abscissa and by $\tau \in I$ time. ■

Furthermore, we assume $\mathcal{L}$ to be a *smooth simple closed curve,* which is equivalent to considering the curve as the image of a differentiable periodic function without singular points or $\Lambda | T$ as a injective map free from self-intersections. It is also useful to introduce the concept of *reparametrization* according to the following.

Definition. U and T are two compact sets in $\mathbb{R}^1$, where T is the domain $D(\Lambda)$. Let Φ be a differentiable (real-valued) function

$$\Phi:\ U \to T\ . \qquad ■$$

Then the composite function

$$\Psi = \Lambda(\Phi):\ U \to \mathcal{M}$$

is a curve called the reparametrization of Λ by Φ. ■

Let us introduce the intrinsic reference frame at any point $\mathbf{X}$ on the curve, invariant with respect to any reparametrization: let $\{\mathbf{e}_k\}$ be the (intrinsic positive) orthonormal reference *Frenet frame* on $\mathcal{L}$. The fundamental existence theorem for space curves in its general form and the extended Frenet-Serret equations as *structure equations* in $\mathcal{M}$ are here recalled.[14]

Theorem. Let $\mathcal{M}$ be a manifold whose affine connection is not necessarily torsion-free. Let

$$\Omega_i^j = \begin{vmatrix} 0 & \Omega_1^2 & 0 & \cdots & 0 & 0 & 0 \\ \Omega_2^1 & 0 & \Omega_2^3 & & 0 & 0 & 0 \\ 0 & \Omega_3^2 & 0 & & 0 & 0 & 0 \\ & & \cdots & \ddots & & & \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & & \Omega_{2n}^{2n-1} & 0 & \Omega_{2n}^{2n+1} \\ 0 & 0 & 0 & \cdots & 0 & \Omega_{2n+1}^{2n} & 0 \end{vmatrix}$$

be the square antisymmetric matrix of $2n$ curvatures $\Omega_j^i(s,\tau)$ such that

$$\Omega_i^j = -\Omega_j^i,\quad i,j=1,\ldots,2n+1$$

where Ω_i^j are $C^{2n},C^{2n-1},\ldots,C^0$ functions of s, $\forall \tau \in I$. Then there exists a curve $\mathcal{L}$ in $\mathcal{M}$ for which the above matrix gives the structure equations as follows:

$$d\mathbf{X}=\mathbf{e}_1 ds,\quad \mathbf{e}_{k/s}=\Omega_k^j \mathbf{e}_j = \Omega_{\mathbf{k}}^{\bar{k}} \mathbf{e}_{\bar{k}}\ , \tag{2.4}$$

with

$$i,k=1,\ldots,2n+1,\quad \bar{k}=k-1,k+1,\quad \forall \mathbf{X}\in\mathcal{L}\ .$$

Such a curve is uniquely determined with respect to a rigid motion in $\mathcal{M}$. □

The usual notation has been adopted for covariant differentiation

$$e^i_{k/s} = \frac{\delta e^i_k}{\delta s} = \frac{\partial e^i_k}{\partial s} + L^i_{jk} e^j_k X^k_s\ ,$$

$$\mathbf{e}_1 = \mathbf{X}_{/s} = \mathbf{X}_s,\quad \mathbf{e}_l = \mathbf{e}_j \delta^j_l,\quad e^k_l = e^j_l g_{ji} g^{lk}\ ,$$

$$\forall \imath \in [1,\ldots,2n+1]\ ,$$

where $\imath$ represents indices. To give the curve physical meaning, it is necessary to specify certain properties. For the sake of concreteness we will discuss string vortices. The choice of the odd dimension of the manifold (and of the ambient space) is primarily due to the required existence of a rotational axis for the curl. This statement is expressed by the following.[15]

Lemma. Let $\mathbf{w}$ be the curl given by $\mathbf{w}=\nabla\times\mathbf{V}$ for every analytic velocity vector field $\mathbf{V}$ in the manifold $\mathcal{M}$. Then we have $\mathbf{e}_1 \neq \mathbf{0}$ by such that

$$\mathbf{w}\cdot\mathbf{e}_1 \times \mathbf{Z} = 0$$

for every arbitrary vector field on $\mathcal{M}$. □

Further, let $\mathbf{w}$ be *nonsingular* in $\mathcal{M}$; thus the direction $\mathbf{e}_1$ is defined uniquely as the direction of the curl of $\mathbf{V}$. We can now assign at any point $\mathbf{X}\in\mathcal{L}$ the vorticity $\mathbf{w}=\bar{w}\cdot\mathbf{e}_1$ everywhere parallel to the tangent vector $\mathbf{e}_1$ to the curve $\mathcal{L}$.

The *material* velocity $\mathbf{V}$ and the *spatial* velocity $\mathbf{v}$ of any point on the vortex string are then defined in their intrinsic components in the usual way,

$$\mathbf{V} = \frac{D\mathbf{X}}{D\tau} = \frac{\delta \mathbf{X}}{\delta s}\frac{\partial s}{\partial \tau}\bigg|_{s_0} \mathbf{e}_1 + \mathbf{v},\quad \mathbf{V} = V^i \mathbf{e}_i\ ,$$

$$\mathbf{X}_{/\tau} = \mathbf{X}_\tau = v^j \mathbf{e}_j = \mathbf{v},\quad \forall \tau \in I\ .$$

The alternative metric a can be introduced considering

$$a_{s,\tau} = g_{ij}\frac{\partial X^i}{\partial s}\frac{\partial X^j}{\partial \tau} = a_{\alpha,\beta},\quad \forall \imath_G \in [s,\tau]$$

(where $\imath_G$ represents greek indices) which requires different quantities of the kind

$$L^i_{jk} = \begin{vmatrix} i \\ jk \end{vmatrix}_g,\quad L^\alpha_{\beta\gamma} = \begin{vmatrix} \alpha \\ \beta\gamma \end{vmatrix}_a\ .$$

For our purposes it will be very useful to use the *Ricci formula* for the commutative covariant (intrinsic) differentiation. In our case this formula becomes simply

$$e^{i}_{k/\alpha\beta}-e^{i}_{k/\beta\alpha}=-e^{j}_{k}R^{i}_{jlm}X^{l}_{\alpha}X^{m}_{\beta}-2e^{i}_{k/\gamma}T^{\gamma}_{\alpha\beta} \tag{2.5}$$

and completes the collection of results we shall need in the following exposition. Before describing in detail the logic of the calculation we used to develop the intrinsic equations for the time evolution of the $2n$ curvatures of the vortex string, we consider the extensibility of $\mathcal{L}$; it is expressed by the relation

$$\mathbf{X}(s+Kds,\tau+d\tau)=\mathbf{X}(s,\tau)+K\frac{\delta\mathbf{X}}{\delta s}ds+\mathbf{v}d\tau\ , \tag{2.6}$$

where K is a constant. As we pointed out in the previous statements, it is convenient to refer to the curve reparametrized by a suitable differentiable (real-valued) function for the transformation of the arc length; without loss of generality this is possible because the term in (2.6)

$$K\frac{\delta\mathbf{X}}{\delta s}ds$$

is contributing along the tangential direction on the curve: it thus has no influence on the motion of the curve, and can be disregarded.

We here sketch the logical scheme of mathematical operations used for deriving the general intrinsic equations. We consider first that the structure equations are given by (2.4) and the velocity of a generic point $\mathbf{X}\in\mathcal{L}$ is given in intrinsic components by $\mathbf{V}[s(s_0,\tau),\tau]$. We then proceed as follows.

(i) We derive $\mathbf{v}$ with respect to the arc length parameter s (covariant differentiation): $\mathbf{v}_{/s}=\mathbf{X}_{\tau s}$.

(ii) We derive $\mathbf{e}_1$ with respect to the time parameter τ (covariant differentiation): $\mathbf{e}_{1/\tau}=\mathbf{X}_{s\tau}$.

(iii) We apply the Ricci formula (2.5) in order to get $\mathbf{e}_{1/\tau}$ as a function of $\mathbf{v}_{/s}$: $\mathbf{e}_{1/\tau}$ is then known.

We do now an iterative process for $h=1,\dots,2n$, that is, until the completion of dimensional space:

(iv) the hth structure equation is used; we derive it with respect to the time parameter (covariant differentiation).

(v) From the previous relation we write $\mathbf{e}_{h+1/\tau}$ as a function of the unknown parameters.

(vi) We apply the Ricci formula (2.5) and calculate $\mathbf{e}_{h/\tau s}$.

(vii) We write $\mathbf{e}_{h/s\tau}$ in explicit form.

(viii) We substitute this value in the relation of point (v) to get $\mathbf{e}_{h+1/\tau}$.

All these operations will be carried out in Sec. III.

III. GENERAL INTRINSIC EQUATIONS OF MOTION

In this section we derive the set of general intrinsic equations governing the motion of the extended one-dimensional classical vortex string in terms of the time derivative of the local curvatures. In order to show how the scheme proposed in Sec. II functions, let us set down the calculation for the case $h=k=1$. Firstly we derive $\mathbf{v}$ covariantly with respect to the arc length parameter s (i),

$$\mathbf{v}_{/s}=\mathbf{X}_{/\tau s}=(v^{k}\mathbf{e}_{k})_{/s},\quad \mathbf{X}_{/\tau s}=v^{k}_{/s}\mathbf{e}_{k}+v^{k}\mathbf{e}_{k/s}\ . \tag{3.1}$$

We derive $\mathbf{e}_1$ covariantly with respect to time τ (ii),

$$\mathbf{e}_{1/\tau}=\mathbf{X}_{/s\tau} \tag{3.2}$$

and by the Ricci formula (2.5), we calculate $\mathbf{e}_{1/\tau}$ as a function of $\mathbf{v}_{/s}$ (iii),

$$\begin{aligned}\mathbf{e}_{1/\tau}&=\mathbf{X}_{/s\tau}=\mathbf{X}_{/\tau s}-2T^{\alpha}_{s\tau}\mathbf{X}_{\alpha}\\&=v^{k}_{/s}\mathbf{e}_{k}+v^{k}\mathbf{e}_{k/s}-2T^{s}_{s\tau}\mathbf{e}_{1}-2T^{\tau}_{s\tau}v^{k}\mathbf{e}_{k}\\&=v^{k}_{/s}\mathbf{e}_{k}+v^{\bar{k}}\Omega^{k}_{\bar{k}}\mathbf{e}_{k}-2T^{\tau}_{s\tau}v^{k}\mathbf{e}_{k}-2T^{s}_{s\tau}\mathbf{e}_{1}\\&=(A^{1}_{1}-2T^{s}_{s\tau})\mathbf{e}_{1}+\cdots+A^{k}_{1}\mathbf{e}_{k}+\cdots+A^{2n+1}_{1}\mathbf{e}_{2n+1}\end{aligned} \tag{3.3}$$

for

$$i,k=1,\dots,2n+1,\quad \bar{k}=k-1,k+1$$

where A^{k}_{1} are functions of $v^{k}_{/s}$ alone

$$A^{k}_{1}=v^{k}_{/s}+v^{\bar{k}}\Omega^{k}_{\bar{k}}-2T^{\tau}_{s\tau}v^{k}\ .$$

Therefore $\mathbf{e}_{1/\tau}$ is known.

We calculate $\mathbf{e}_{2/\tau}$; the first structure equation [Eq. (2.4), $h=k=1$], when it is derived covariantly with respect to the time parameter τ, yields (iv)

$$(\mathbf{e}_{1/s})_{/\tau}=(\Omega^{j}_{1}\mathbf{e}_{j})_{/\tau}=\Omega^{j}_{1/\tau}\mathbf{e}_{j}+\Omega^{j}_{1}\mathbf{e}_{j/\tau}\ .$$

We obtain $\mathbf{e}_{2/\tau}$ as a function of the unknown parameters (v),

$$\mathbf{e}_{1/s\tau}=\Omega^{j}_{1/\tau}\mathbf{e}_{j}+\Omega^{2}_{1}\mathbf{e}_{2/\tau},\quad \Omega^{2}_{1}\mathbf{e}_{2/\tau}=\mathbf{e}_{1/s\tau}-\Omega^{2}_{1/\tau}\mathbf{e}_{2}$$

and applying the Ricci formula (2.5) once more (vi),

$$\begin{aligned}e^{i}_{1/s\tau}-e^{i}_{1/\tau s}&=-R^{i}_{lhm}X^{h}_{s}X^{m}_{\tau}e^{l}_{1}-2T^{\alpha}_{s\tau}e^{i}_{1/\alpha}\\&=-R^{l}_{lhm}v^{h}\delta^{m}_{1}e^{i}_{1}-2T^{s}_{s\tau}e^{i}_{1/s}-2T^{\tau}_{s\tau}e^{i}_{1/\tau}\ .\end{aligned}$$

We write $\mathbf{e}_{1/s\tau}$ in explicit from (vii),

$$\begin{aligned}\mathbf{e}_{1/s\tau}=&[(A^{1}_{1}-2T^{s}_{s\tau})_{/s}\mathbf{e}_{1}+\cdots+A^{k}_{1/s}\mathbf{e}_{k}+\cdots]\\&+(A^{1}_{1}-2T^{s}_{s\tau})\Omega^{2}_{1}\mathbf{e}_{2}+\cdots+A^{\bar{k}}_{1}\Omega^{k}_{\bar{k}}\mathbf{e}_{k}+\cdots\\&-R^{l}_{lhm}v^{h}\delta^{m}_{1}\mathbf{e}_{1}-2T^{s}_{s\tau}\Omega^{2}_{1}\mathbf{e}_{2}-2T^{\tau}_{s\tau}\mathbf{e}_{1/\tau}\ .\end{aligned}$$

Finally, inserting relation (vii) into (v) we obtain $\mathbf{e}_{2/\tau}$ in the form (viii)

$$\begin{aligned}\Omega^{2}_{1}\mathbf{e}^{i}_{2/\tau}=&[(A^{1}_{1}-2T^{s}_{s\tau})_{/s}+A^{2}_{1}\Omega^{1}_{2}-R^{l}_{lhm}v^{h}\delta^{m}_{1}-2T^{\tau}_{s\tau}(A^{1}_{1}-2T^{s}_{s\tau})]\mathbf{e}_{1}\\&+(A^{2}_{1/s}+A^{1}_{1}\Omega^{2}_{1}+A^{3}_{1}\Omega^{2}_{3}-2T^{\tau}_{s\tau}A^{2}_{1}-\Omega^{2}_{1/\tau}-4T^{s}_{s/\tau}\Omega^{2}_{1})\mathbf{e}_{2}+\cdots+(A^{k}_{1/s}+A^{\bar{k}}_{1}\Omega^{k}_{\bar{k}}-2T^{\tau}_{s\tau}A^{k}_{1})\mathbf{e}_{k}+\cdots\ .\end{aligned}$$

Explicitly, $\mathbf{e}_{2/\tau}$ is given by

$$\mathbf{e}_{2/\tau}=(A_2^1+\overline{A}_2^1)\mathbf{e}_1+\left[A_2^2-\frac{\Omega_{1/\tau}^2}{\Omega_1^2}-4T_{s\tau}^s\right]\mathbf{e}_2 + \cdots + A_2^k\mathbf{e}_k+\cdots+A_2^{2n+1}\mathbf{e}_{2n+1} , \tag{3.4}$$

where we have indicated the known quantities in the compact form

$$\Omega_1^2 A_2^k = A_{1/s}^k + A_1^{\bar{k}}\Omega_{\bar{k}}^k - 2T_{s\tau}^\tau A_1^k ,$$

$$\Omega_1^2 \overline{A}_2^1 = (-2T_{s\tau}^s)_{/s} - R_{lhm}^l v^h \delta_1^m + 4T_{s\tau}^\tau T_{s\tau}^s ,$$

$$\Omega_1^2 \overline{A}_2^2 = -\Omega_{1/\tau}^2 - 4T_{s\tau}^s \Omega_1^2 .$$

In the same way we shall repeat the above calculation so as to derive $\mathbf{e}_{3/\tau}$, $\mathbf{e}_{4/\tau}$, etc. After some straightforward but tedious algebra we find the general relation for $\mathbf{e}_{j/\tau}$,

$$\mathbf{e}_{j/\tau}=(A_j^1+\overline{A}_j^1)\mathbf{e}_1+\cdots+(A_j^j+\overline{A}_j^j)\mathbf{e}_j+A_j^{j+1}\mathbf{e}_{j+1} + \cdots + A_j^k\mathbf{e}_k+\cdots+A_j^{2n+1}\mathbf{e}_{2n+1} \tag{3.5}$$

with the standard notation

$$\Omega_{j-1}^j A_j^k = A_{(j-1)/s}^k + A_{j-1}^{\bar{k}}\Omega_{\bar{k}}^k - 2T_{s\tau}^\tau A_{j-1}^k - \Omega_{j-1}^{j-2} A_{j-2}^k ,$$

$$\Omega_{j-1}^j \overline{A}_j^l = \overline{A}_{(j-1)/s}^l + \overline{A}_{j-1}^{\bar{l}}\Omega_{\bar{l}}^l - 2T_{s\tau}^\tau \overline{A}_{j-1}^l - \Omega_{j-1}^{j-2}\overline{A}_{j-2}^l ,$$

$$\Omega_{j-1}^j \overline{A}_j^{j-2} = \Omega_{j-1}^j \overline{A}_j^{l+1} - 2T_{s\tau}^s \Omega_{j-1}^{j-2} - \Omega_{(j-1)/\tau}^{j-2} ,$$

$$\Omega_{j-1}^j \overline{A}_j^{j-1} = \overline{A}_{(j-1)/s} + \overline{A}_{j-1}^{j-2}\Omega_{j-2}^{j-1} - R_{lhm}^l v^h \delta_1^m - 2T_{s\tau}^\tau \overline{A}_{j-1}^{j-1} ,$$

$$\Omega_{j-1}^j \overline{A}_j^j = -\sum_{h=1}^{j-2} \frac{\Omega_{h/\tau}^{h+1}}{\Omega_h^{h+1}}\Omega_{j-1}^j - \Omega_{(j-1)/\tau}^j - 2jT_{s\tau}^s ,$$

for

$$k=1,\ldots,2n+1,\quad l=1,\ldots,j-3 ,$$

$$\bar{k}=k+1,k-1,\quad \bar{l}=l+1,l-1 .$$

We now differentiate the orthonormal basis $\{\mathbf{e}_k\}$ with respect to time

$$\frac{D\mathbf{e}_k}{D\tau}=\frac{\delta\mathbf{e}_k}{\delta s}\bigg|_\tau \frac{\partial s}{\partial\tau}\bigg|_{s_0}+\frac{\delta\mathbf{e}_k}{\delta\tau}\bigg|_s=\Omega_k^{\bar{k}}\mathbf{e}_{\bar{k}}\partial_\tau s|_{s_0}+\mathbf{e}_{k/\tau} \tag{3.6}$$

and by the orthogonality condition for the orthonormal basis we write

$$\frac{D\mathbf{e}_j}{D\tau}\cdot\mathbf{e}_k=-\frac{D\mathbf{e}_k}{D\tau}\cdot\mathbf{e}_j ,$$

$$\frac{D}{D\tau}(\Omega_j^k\mathbf{e}_k)\cdot\mathbf{e}_k=0$$

putting the relations (3.6) in the following matrix form:

$$\frac{D}{D\tau}\begin{bmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \vdots\\ \mathbf{e}_{2n+1}\end{bmatrix}=\begin{bmatrix}B_{1,1} & B_{1,2} & \cdots & B_{1,2n+1}\\ B_{2,1} & B_{2,2} & & B_{2,2n+1}\\ \vdots & & \ddots & \vdots\\ B_{2n+1,1} & B_{2n+1,2} & \cdots & B_{2n+1,2n+1}\end{bmatrix}\begin{bmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \vdots\\ \mathbf{e}_{2n+1}\end{bmatrix} ,$$

where

$$B_{j,k}=-B_{k,j},\quad B_{k,k}=0 .$$

In detail $B_{j,k}$ are given by

$$B_{j,k}=-B_{k,j} ,$$

$$B_{j,k}=\Omega_j^{j+1}\partial_\tau s|_{s_0} + \frac{A_{(j-1)/s}^k + A_{j-1}^{\bar{k}}\Omega_{\bar{k}}^k - 2T_{s\tau}^\tau A_{j-1}^k - \Omega_{j-1}^{j-2}A_{j-2}^k}{\Omega_{j-1}^j} , \tag{3.7}$$

$$B_{k,k}=0 ,$$

with

$$A_j^k=\frac{A_{(j-1)/s}^k + A_{j-1}^{\bar{k}}\Omega_{\bar{k}}^k - 2T_{s\tau}^\tau A_{j-1}^k - \Omega_{j-1}^{j-2}A_{j-2}^k}{\Omega_{j-1}^j} .$$

The third condition of (3.7), namely

$$B_{k,k}=0,\quad k=1,\ldots,2n+1$$

gives for $k=1$ the congruence equation and for $k=2,\ldots,2n+1$ the $2n$ intrinsic equations of motion in $\mathcal{M}$. The proof of the first statement is given by considering two close points $\mathbf{X}_1$ and $\mathbf{X}_2$ on $\mathcal{L}$ such that

$$\mathbf{X}_2-\mathbf{X}_1=\mathbf{e}_1 ds$$

and also

$$(\mathbf{X}_2)_{/\tau}=\mathbf{v}+(\mathbf{X}_1)_{/s\tau}ds,\quad (\mathbf{X}_1)_{/\tau}=\mathbf{v} .$$

Then, writing the elementary distance ds as

$$[(ds)^2]_{/\tau}=2(\mathbf{X}_2-\mathbf{X}_1)_{/\tau}\cdot(\mathbf{X}_2-\mathbf{X}_1) =2(\mathbf{X}_1)_{/s\tau}\cdot\mathbf{e}_1(ds)^2 \tag{3.8}$$

and making $B_{1,1}=0$ explicit, we prove by the Ricci formula (2.5) the congruence condition for the material points of the curve, that is, its inextensibility ($ds=0$).

We prove the second statement, given by the condition

$$B_{k,k}=A_k^k-\sum_{i=1}^{k-1}\frac{\Omega_{i/\tau}^{i+1}}{\Omega_i^{i+1}}-2kT_{s\tau}^s=0 \tag{3.9}$$

and rewritten as

$$A_k^k-\frac{\Omega_{(k-1)/\tau}^k}{\Omega_{k-1}^k}-\sum_{i=1}^{k-2}\frac{\Omega_{i/\tau}^{i+1}}{\Omega_i^{i+1}}-2kT_{s\tau}^s=0 ,$$

by recalling the definition of the velocity as stated in the Sec. II with the notation

$$V^1=\partial_\tau s|_{s_0}+v^1, \quad v^1=V^1-\partial_\tau s|_{s_0}$$

and by calculating the equations taking the time derivative of the $(k-1)$ curvature as follows:

$$\frac{D}{D\tau}(\Omega_{k-1}^k)=\frac{\delta\Omega_{k-1}^k}{\delta s}\bigg|_\tau\frac{\partial s}{\partial\tau}\bigg|_{s_0}+\frac{\delta\Omega_{k-1}^k}{\delta\tau}\bigg|_s =\Omega_{(k-1)/s}^k\partial_\tau s|_{s_0}+\Omega_{(k-1)/\tau}^k .$$

Thus, the complete set of equations in their Lagrangian form is given by

$$\begin{aligned}(V^1-\partial_\tau s|_{s_0})_{/s}&=2T_{s\tau}^\tau v^1-\Omega_2^1 v^2+2T_{s\tau}^s \quad \text{for } k=1 ,\\ \frac{D}{D\tau}(\Omega_{k-1}^k)&=\Omega_{(k-1)/s}^k\partial_\tau s|_{s_0}\\ &\quad+\Omega_{k-1}^k\left[A_k^k-\sum_{i=1}^{k-2}\frac{\Omega_{i/\tau}^{i+1}}{\Omega_i^{i+1}}-2kT_{s\tau}^s\right] ,\end{aligned} \tag{3.10}$$

where we have written for $k=1$ the *congruence equation* and for $k=2,\ldots,2n+1$ the $2n$ *intrinsic equations* of motion in index form; they express indeed the time derivative of the curvatures in terms of the intrinsic components of the velocity. The set of equations governs the global kinematics of the extended one-dimensional classical vortex string $\mathcal{L}$ instantaneously embedded in the manifold $\mathcal{M}$, whose connection we have not assumed to be torsion-free.

IV. PARTICULAR CASES OF SPECIAL INTEREST

Equations (3.10), derived in Sec. III, have been obtained by a general induction process and they conserve the structural regularity due to the particular antisymmetric curvature matrix associated with the general Frenet-Serret formulas (2.4). In other words, the extended equations (2.4) that preserve their regular background structure in higher-dimensional spaces induce the same structural regularity in the intrinsic equations (3.10): in particular, we observe the influence of the curvature variation in one specified direction upon the curvature variation in that direction in a higher-dimensional order. This is due to the role of the structure equations and to the fundamental bearing they have on the framework of the logical scheme of derivation. Let us consider now in some detail particular cases of special interest.

A. The symmetric affine connection case

In this case the torsion tensor is null, and the set of equations (3.10) assumes the simplified form

$$\begin{aligned}(V^1-\partial_\tau s|_{s_0})_{/s}&=-\Omega_2^1 v^2 \quad \text{for } k=1 ,\\ \frac{D}{D\tau}(\Omega_{k-1}^k)&=\Omega_{(k-1)/s}^k/\partial_\tau s|_{s_0}+\Omega_{k-1}^k\left[\frac{A_{(k-1)/s}^k+A_{k-1}^{\bar{k}}\Omega_{\bar{k}}^k-\Omega_{k-1}^{k-2}A_{k-2}^k}{\Omega_{k-1}^k}-\sum_{i=1}^{k-2}\frac{\Omega_{i/\tau}^{(i+1)}}{\Omega_i^{i+1}}\right] ,\end{aligned} \tag{4.1}$$

where, as before, $k=1$ gives the congruence equation and $k=2,\ldots,2n+1$ the $2n$ intrinsic equations.

B. The Riemannian connection case

Since in this case we have a symmetric connection, the equations are formally identical to (4.1); the covariant differentiation involved here is carried out using the Christoffel symbols of the second kind—usually denoted as Γ_{jk}^i—instead of L_{jk}^i.

C. The three-dimensional Euclidean case

We briefly reconsider the simple three-dimensional Euclidean case in order to show one of the motivations of our analysis. For this purpose we recall the set of equations firstly derived by Germano[1] in the form

$$\begin{aligned}v_t'&=v_n c ,\\ \frac{\partial c}{\partial t}\bigg|_s&=(v_n'+v_t c-v_b\theta)'-(v_b'+v_n\theta)\theta ,\\ \frac{\partial\theta}{\partial\tau}\bigg|_s&=\left[\frac{(v_n'+v_t c-v_b\theta)\theta+(v_b'+v_n\theta)'}{c}\right]'\\ &\quad+(v_b'+v_n\theta)c ,\end{aligned} \tag{4.2}$$

where we denote the curvature by c and the torsion by θ and write

$$\partial_s|_\tau=(\ )' ,$$

and where we have used the standard notation for the velocity

$$\mathbf{v}=v_t\mathbf{t}+v_n\mathbf{n}+v_b\mathbf{b}$$

with respect to the Frenet-Serret reference frame $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$.

Assuming the vortex line is subjected to the so-called local induction approximation[16] (LIA), we have an even simpler self-induced law of motion, such as

$$\mathbf{v}=v_b\mathbf{b}=c\mathbf{b}\ ,$$

which leads to the set of equations obtained by Da Rios[3] and Betchov[4] independently and rediscovered in a different context several years later.[17]

Hasimoto,[18] combining together the curvature and the torsion of the filament into one complex variable ψ, in the form

$$\psi(s,\tau)=c(s,\tau)\exp\left[i\oint\theta(s,\tau)ds\right],\quad \psi(s,\tau)\in\mathbb{C}$$

has shown that these equations can be reduced to the nonlinear Schrödinger equation (NLSE); this link was successively extended for several different nonlinear dynamical equations.[19,20] In particular, it has been shown that there is a connection between certain types of helical space curves and the sine-Gordon and the Hirota equations (and consequently the nonlinear Schrödinger and the modified Korteweg–de Vries equations), a connection that occurs for different realization of the intrinsic components of the velocity of the string. It has also been shown how, by generalizing the local induction equation by physical arguments, to include inhomogeneities and dissipative effects.[21,22] Furthermore, since the latter equations are completely integrable and their solutions are expressible in terms of soliton solutions,[23,24] it is also evident that there exists an infinite set of invariants, namely the conservation laws for the associated nonlinear equation,[25,26] which emerge from the study of the geometrical spatial properties of the general intrinsic equations. For example, in the case of NLSE we can easily apply the scheme proposed by Zakharov and Shabat[27] to calculate the conserved quantities in terms of intrinsic parameters. They are constants of motion, and represent constraints on the kinematical behavior of the spatial vortex line.

V. CONCLUDING REMARKS

By an algebraic-geometrical generalization we have obtained the intrinsic equations governing the time evolution of the curvatures of an extended isolated classical vortex string moving in an unbounded, perfect fluid flow. The filament is instantaneously embedded in a curved manifold, whose connection we are not assuming to be torsion-free, and which can be assumed to be locally homeomorphic to an odd-dimensional Euclidean space. This general case is interesting because the relevant group is that of volume-preserving diffeomorphisms of the domain of fluid flow, and because the principle of least action implies that the motion of the fluid is described by the geodesics in the metric given by the kinetic energy. Considering the fluid in the ambient space under the condition of divergence, we have suggested how to deal with a solenoidal field by geometrizing the compressibility. Since the mapping of the compressibility into the divergence field is given by the Jacobian, we have constructed a metric and a connection of the manifold, both of them also functions of the Jacobian.

We have discussed some particular cases of special interest, such as the torsion-free affine connection case and the Riemannian connection case, mentioning, in three-dimensional Euclidean space, the linking of the particular set of equations (the Da Rios–Betchov equations) with the nonlinear Schrödinger equation (NLSE) by the Hasimoto (gauge) transformation.

The analysis presented here suggests the possibility of extending such a linking to a more general set of nonlinear equations. In this context it is clear that our general study of intrinsic equations should help to bring these links into evidence, in order to establish a more organic and deeper understanding of the connections between the kinematics of curves and the general properties of soliton equations.

Moreover, given that the vortex string is a generator of the corresponding soliton surface, it can be seen as a geodesic on that surface;[28,29] a study of the intrinsic characteristics of the generative string will enhance our understanding of the topological properties of the surface. We pointed out, for example, how in the case of NLSE we can write the conservation laws in terms of intrinsic parameters of the curve and then interpret the constants of motion as geometrical constraints. Though the role played by these invariants is still to be understood, the existence of a general set which could be inclusive of the specific ones is obviously connected with the existence of a more general soliton equation, according to an appropriate transformation for the intrinsic equations that we have here deduced. Furthermore, this formulation also provides a newer approach to one classical problem in geometry, the so-called plateau problem. Since the motion of the filament is governed by the principle of least action,[30,31] then the string-generated surface clearly has a minimum area. As a result we are led to more abstract ways of studying minimal surfaces on manifolds. This research, then, seems to us to have wide implications; work is in progress along these lines.

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