Lecture 6

- ***Topological decay measured by knot polynomials:***

  - ***Knot polynomials as new tools in turbulence research***
  - ***Polynomial skein relations***
  - ***Adapted polynomials as new invariants of fluid mechanics***
  - ***Vortex knot cascade detected by monotonically decreasing sequence of polynomial numerical values***
  - ***HOMFLYPT as best quantifier of topological complexity***

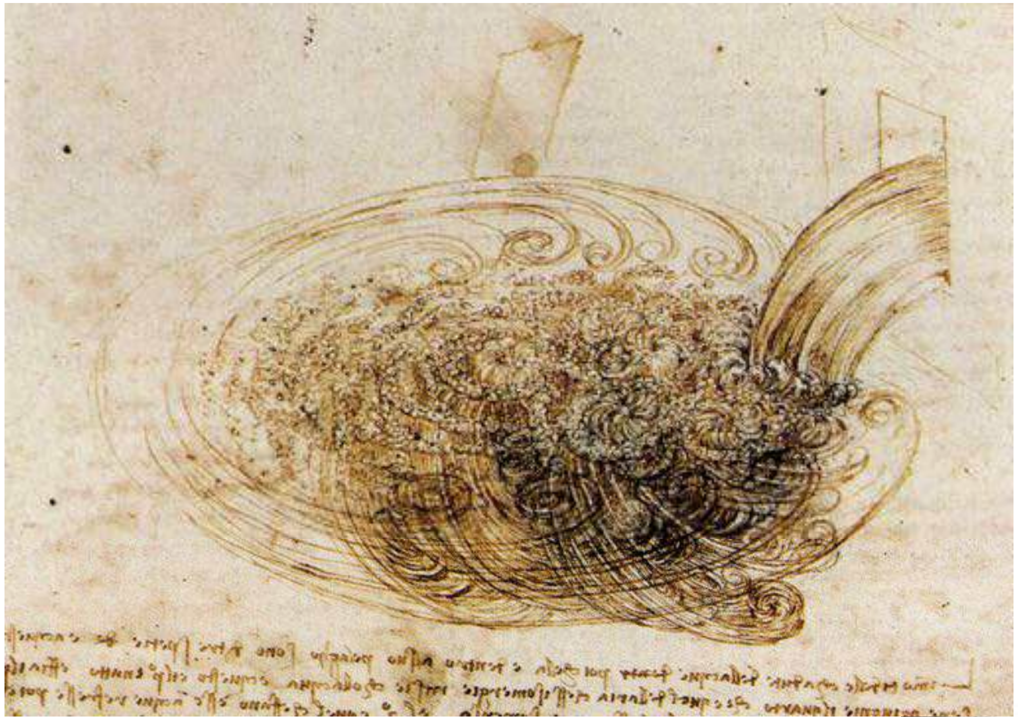
- ***Selected references***

Liu, X & Ricca, RL 2012 ***The Jones polynomial for fluid knots from helicity.*** *J Phys A: Math & Theor*, **45, 205501.**

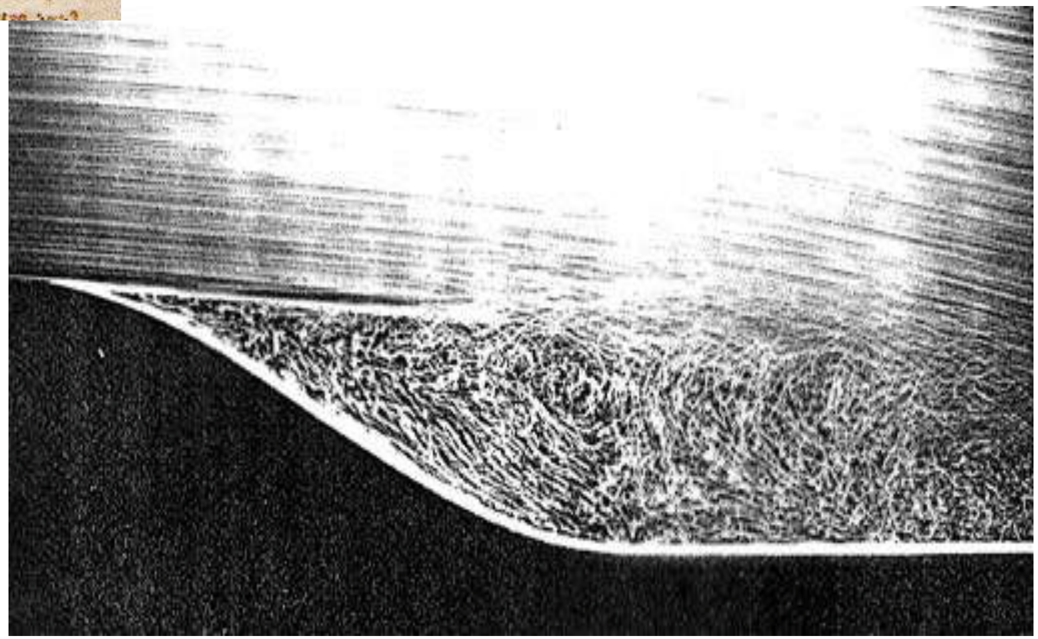
Liu, X & Ricca, RL 2015 ***On the derivation of HOMFLYPT polynomial invariant for fluid knots.*** *J Fluid Mech* **773, 34-48.**

Liu, X & Ricca, RL 2016 ***Knots cascade detected by a monotonically decreasing sequence of values.*** *Nature Sci Rep* **6, 24118.**

Coherent structures

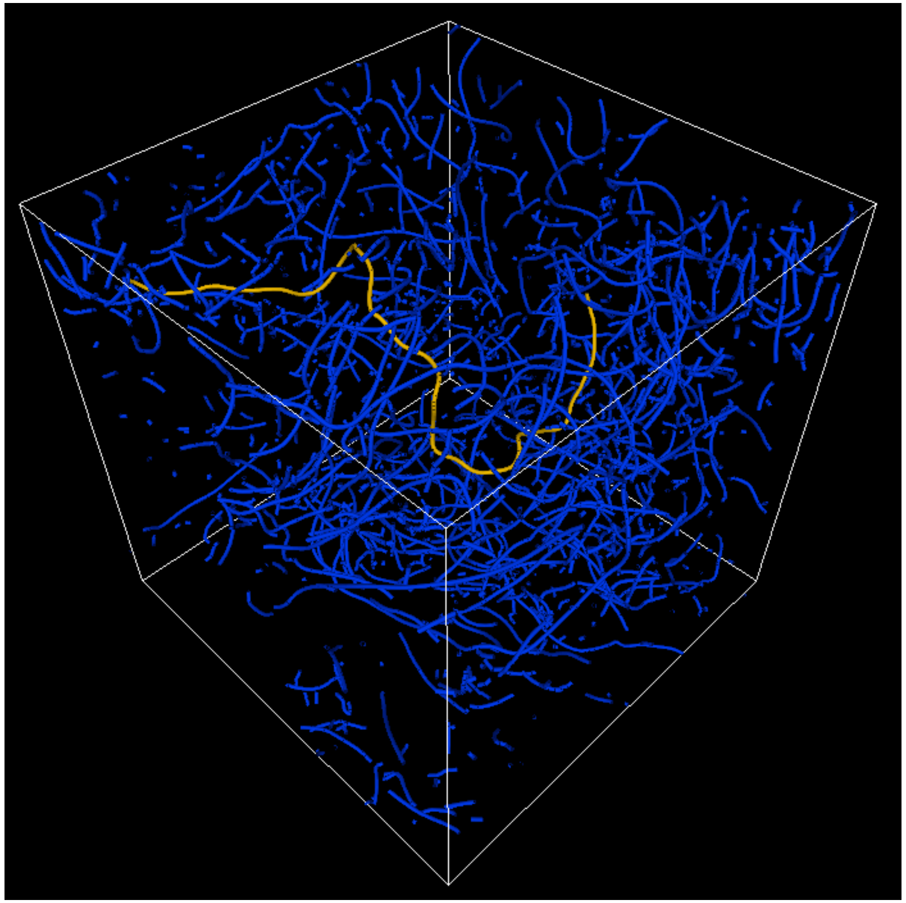


(Leonardo da Vinci 1506)

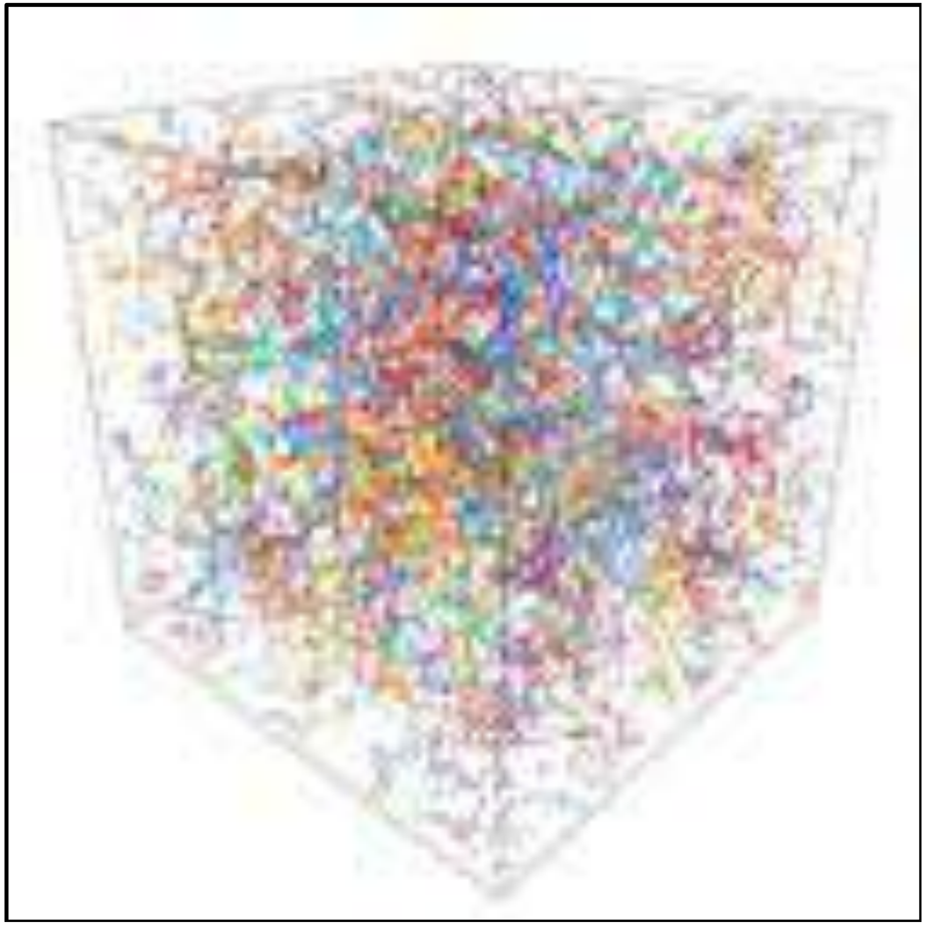


(Werlé 1974)

Vorticity localization in classical and quantum fluids

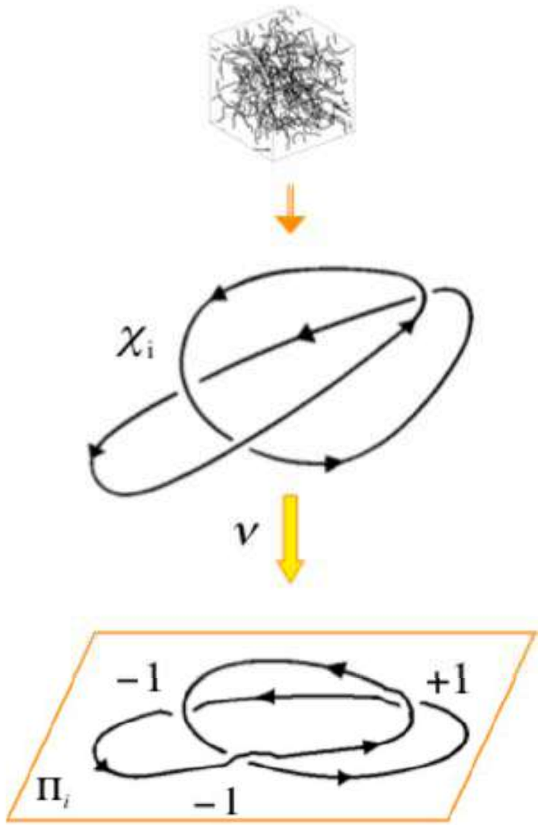


(Kida et al. 2002)



(Villois et al. 2016)

Tackling structural complexity by knot polynomials



- ***Helicity and linking number limitations:***

(i) $$H(K) = f\left(SL_i, Lk_{ij}; \Gamma\right)$$

(ii) $$Lk_{ij} = 0\,, \quad \sum_{i \neq j} Lk_{ij} = 0\,.$$

- ***HOMFLYPT polynomial*** $P(K) = P_K(a,z)$ **:**

(P.1) $$P(\mathrm{O}) = 1$$

(P.2) $aP\Big($  $\Big) - a^{-1}P\Big($  $\Big) = zP\Big($  $\Big)$

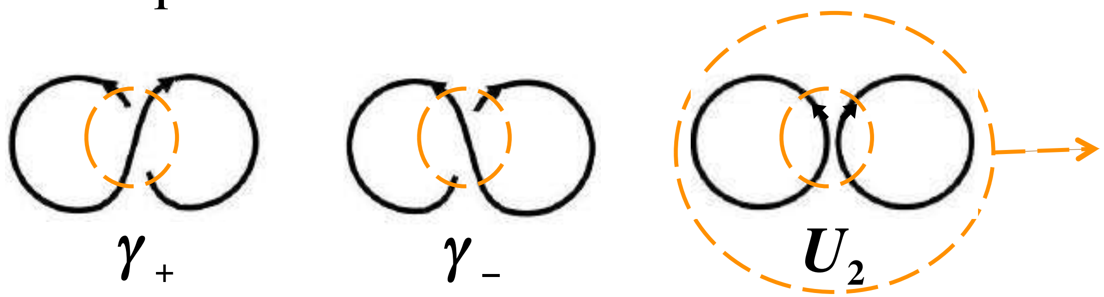
$+$ $\quad$ $-$ $\quad$ $=$

P.1:

$\boldsymbol{U}_1$ $\quad$ γ_+ $\quad$ γ_-

$$P(\mathrm{O}) = P(\gamma_+) = P(\gamma_-) = 1$$

P.2:

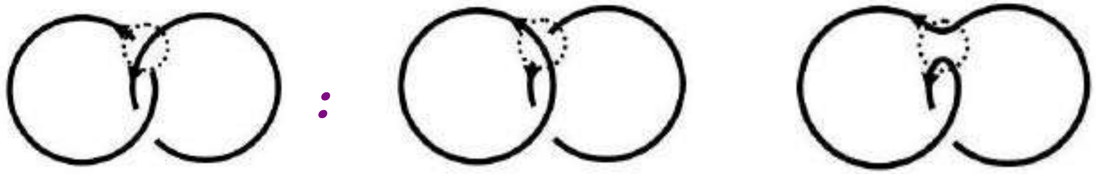


$$P(\boldsymbol{U}_2) = \frac{a - a^{-1}}{z}$$

Knot polynomials: examples

- ***Hopf link H_- :***

$$P(H_-) = -az-(a-a^3)z^{-1}$$



H_- $\quad$ l_{cc} $\quad$ γ_-

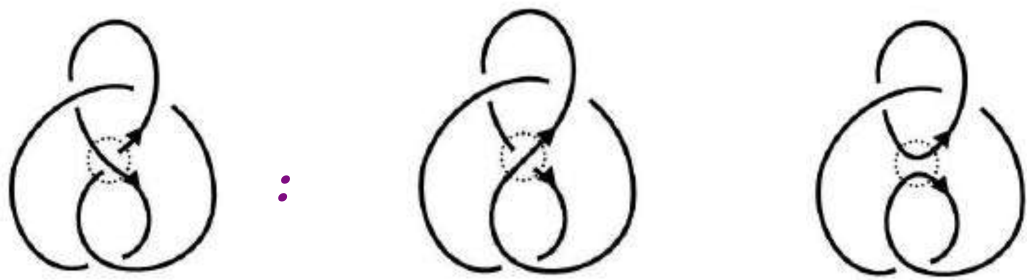
- ***Left-handed trefoil knot T^L :***

$$P(T^L) = 2a^2 + a^2z^2 - a^4$$

T^L $\quad$ ***unknot*** $\quad$ H_-

- ***Figure-of-eight knot F^8 :***

$$P(F^8) = a^{-2} - 1 - z^2 + a^2$$



F^8 $\quad$ ***unknot*** $\quad$ H_-

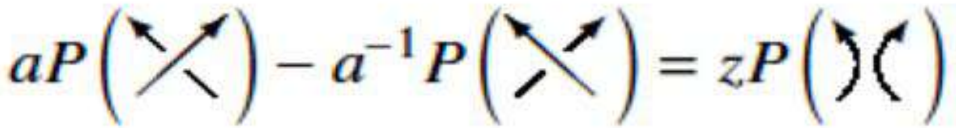
HOMFLYPT polynomial from self-linking

- **Theorem (Liu & Ricca 2015).** *If K denotes a vortex knot of axis χ and helicity $H = H(K)$, then*

$$e^{Lk(K)} = e^{\oint_K \boldsymbol{u}\cdot d\boldsymbol{l}} ,$$

appropriately rescaled, satisfies (with a plausible statistical hypothesis) the skein relations of the HOMFLYPT polynomial $P(K) = P_K(a,z)$.

- **HOMFLYPT variables in terms of writhe and twist:**



$$\Rightarrow \quad [f(Tw)] = g(Wr) .$$

with $z = k - k^{-1}$ **and**

$$\begin{cases} k = e^{2\omega}, & \omega = \lambda_\omega \langle Wr \rangle \\ a = e^{\tau}, & \tau = \lambda_\tau \langle Tw \rangle \end{cases} \quad \text{and} \quad \{\lambda_\omega, \lambda_\tau\} \in (0;1) ,$$

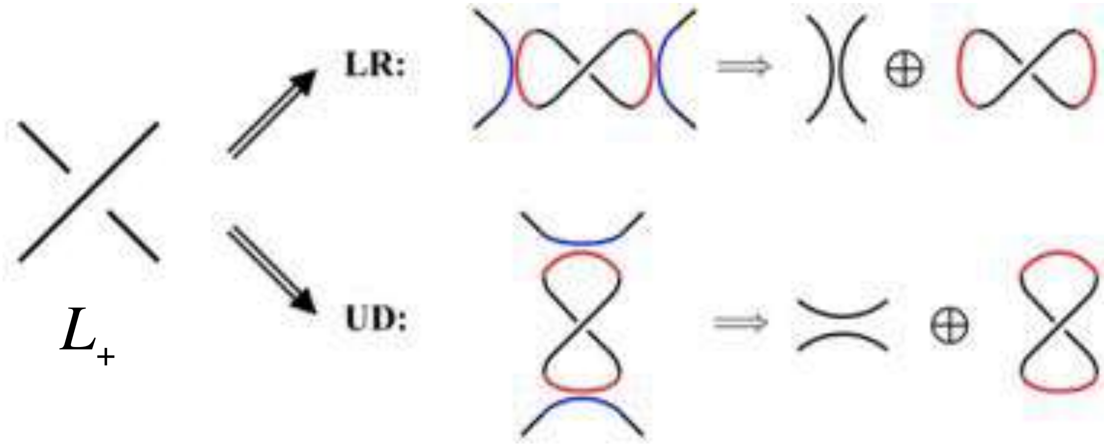
hence $a = f(Tw)$ **and** $z = g(Wr)$.

- **Reduction of HOMFLYPT to Jones:**

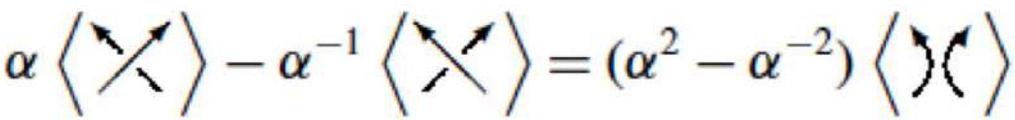
$$ak^2 = e^{\tau}e^{4\omega} = 1 \quad \Rightarrow \quad Wr = -4\lambda\, Tw \qquad (\lambda = \lambda_\tau / \lambda_\omega) .$$

Proof.

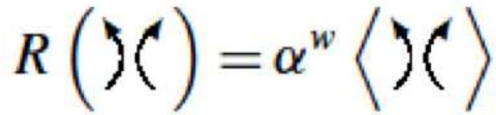
- *derive the Kauffman bracket* $\langle \cdot \rangle$ *polynomial for unoriented knot; assume equal probability in state decomposition:*



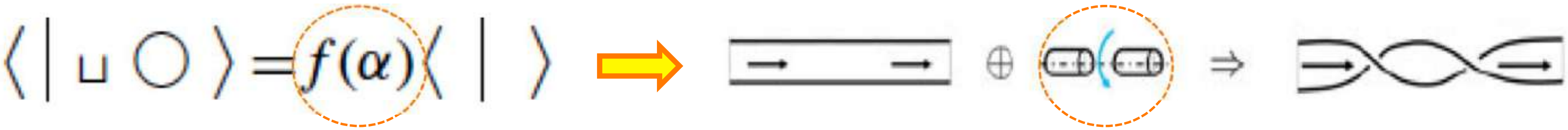
- *orient knot*
- *derive skein relation for* z *in terms of* Wr, *considering*



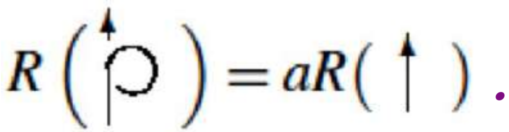
and



- *note*



- *derive skein relation for* a *in terms of twist* Tw, *considering*



∎

Quantifying topological complexity

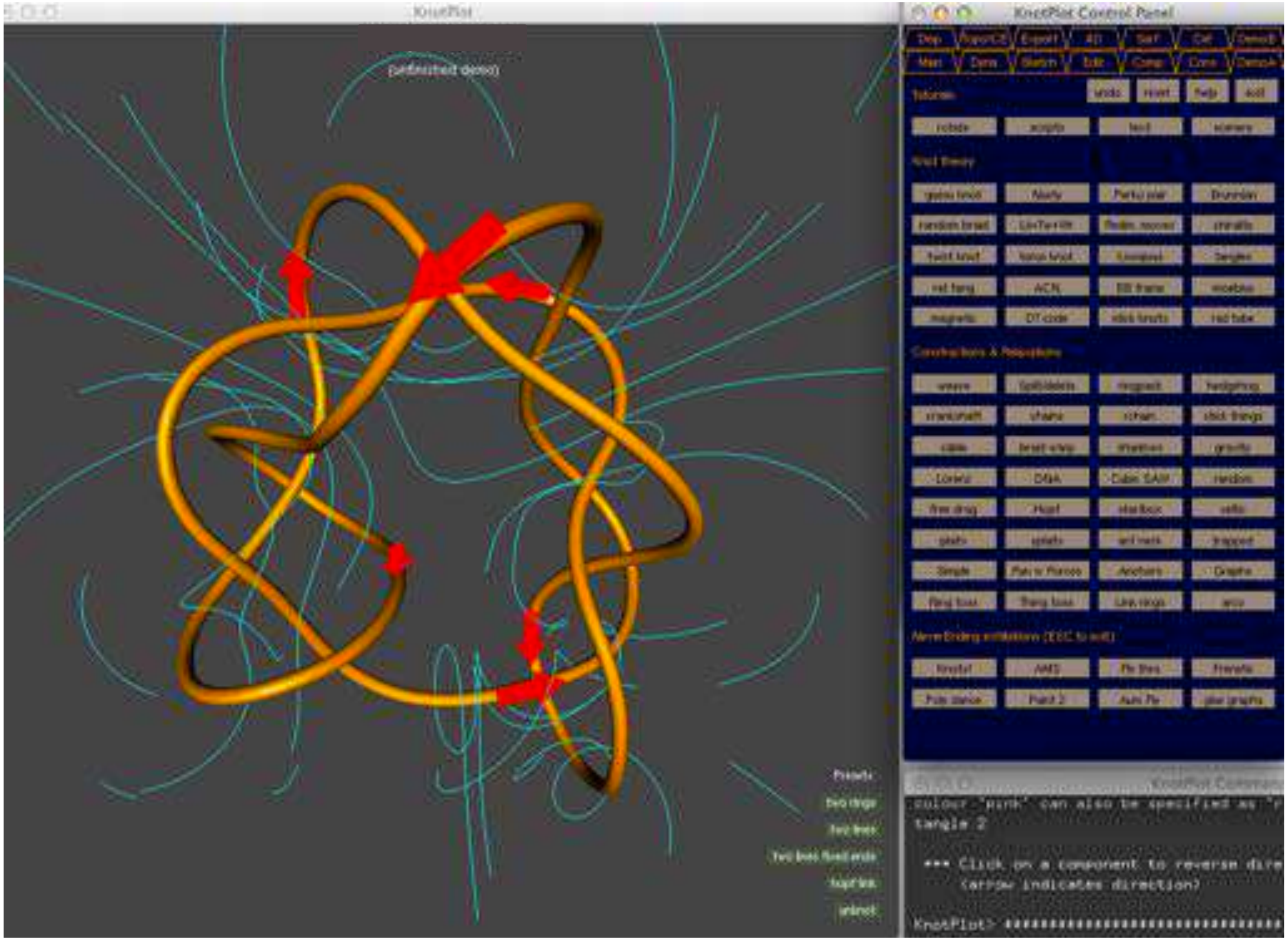
In general we shall have $P(K) = f(K, \Gamma)$.

- *Homogeneous superfluid tangle:* $\Gamma = 1$ *and*

$$\begin{cases} k = e^{2\omega}, & \omega = \lambda_\omega \langle Wr \rangle \\ a = e^{\tau}, & \tau = \lambda_\tau \langle Tw \rangle \end{cases} \quad \textit{with} \quad \begin{matrix} \langle Wr \rangle = \langle Tw \rangle = 1/2 \\ \lambda_\omega = \lambda_\tau = 1/2 \end{matrix} \quad \Rightarrow \quad \begin{cases} z = e^{1/2} - e^{-1/2} \\ a = e^{1/4} \end{cases}$$

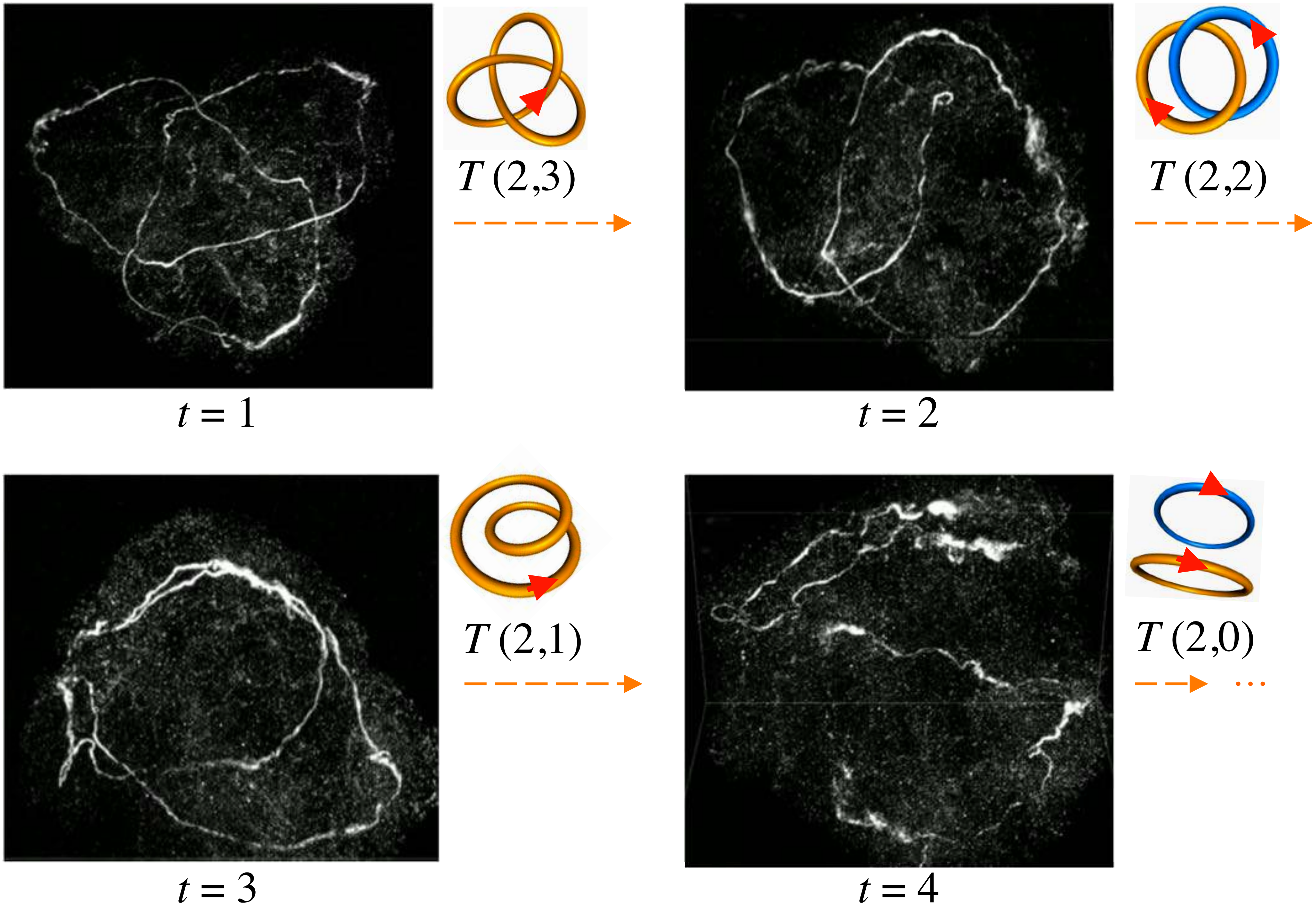
Knot type	HOMFLYPT polynomial	Numerical value
U_N	$\delta^{N-1} = [(a - a^{-1})z^{-1}]^{N-1}$	0.48^{N-1}
H_+	$a^{-1}z + (a^{-1} - a^{-3})z^{-1}$	1.10
H_-	$-az - (a - a^3)z^{-1}$	−0.54
T^L	$2a^2 + a^2z^2 - a^4$	2.36
T^R	$2a^{-2} + a^{-2}z^2 - a^{-4}$	1.51
F^8	$a^{-2} - 1 - z^2 + a^2$	0.17
W	$-a^{-1}z^{-1} - a^{-1}z + az^{-1} + 2az + az^3 - a^3z$	1.59
…	…	…

Topological diagnostics by visiometrics

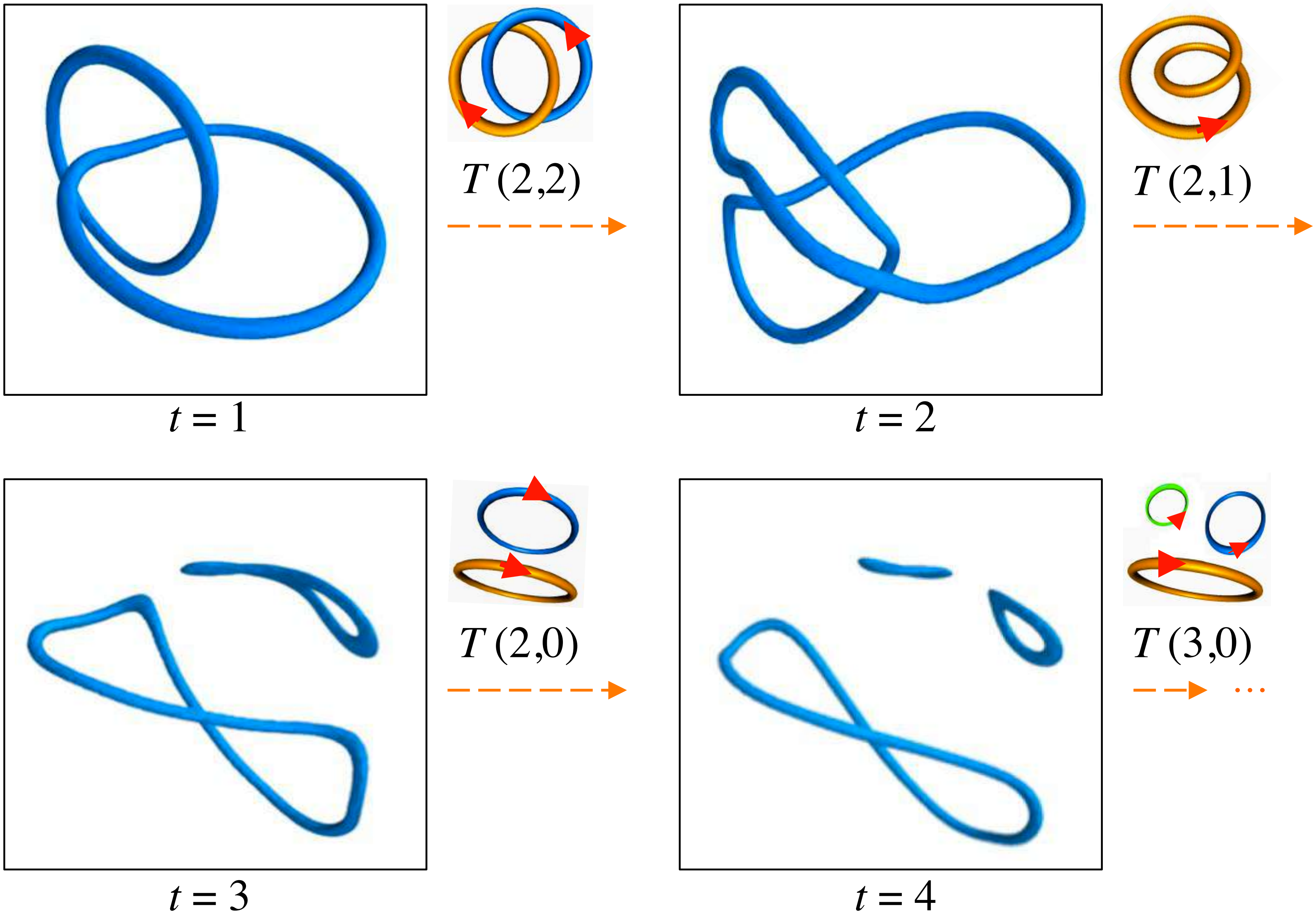


KnotPlot *(Scharein 2000)*

Vortex trefoil cascade process in water (Kleckner & Irvine 2013)



Vortex link cascade in BECs (Zuccher & Ricca 2017)

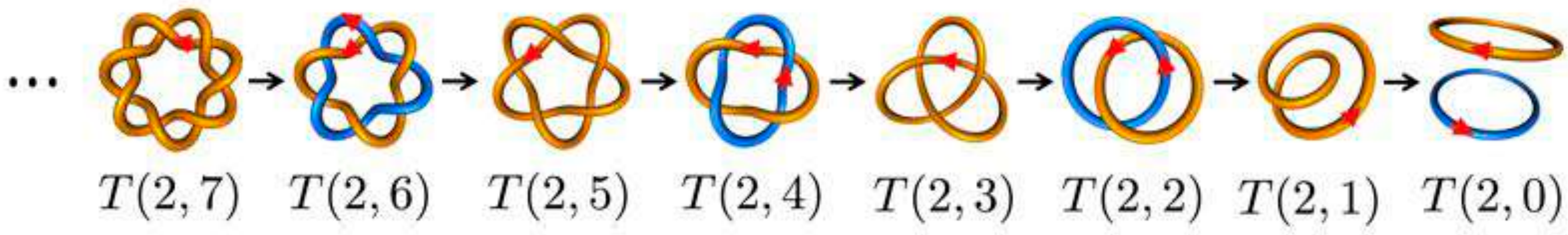


Ideal torus knots & links cascade

Consider the cascade process:

(i)

(ii)



$$\{T(2,n)\}: \quad \ldots \rightarrow T(2,2n+1) \rightarrow T(2,n) \rightarrow \ldots \rightarrow T(2,0)\ .$$

Assumptions:

- ***all torus knots** $T(2,2n+1)$ **and links** $T(2,2n)$ **are standardly embedded on a mathematical torus in closed braid form;***
- ***all torus knots and links form an ordered set** $\{T(2,n)\}$ **of elements listed according to their decreasing value of topological complexity given by** $c_{\min} = n$ **;***
- ***any topological transition between two contiguous elements of** $\{T(2,n)\}$ **is determined by a single, orientation-preserving reconnection event.***

Torus knots cascade detected by HOMFLYPT

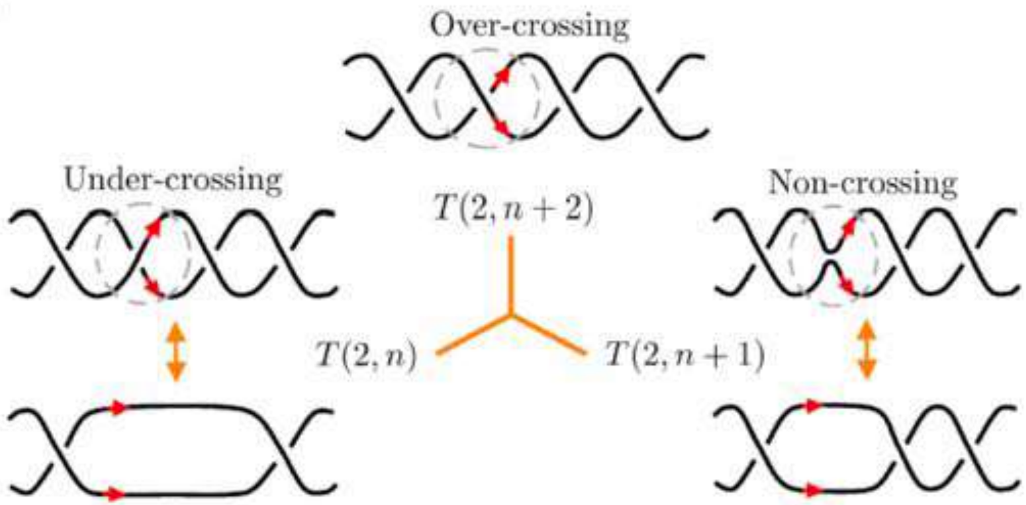
- **Theorem (Liu & Ricca 2016).** *HOMFLYPT computation of* $P_{T(2,n)}$ *generates, for decreasing* n*, a monotonically decreasing sequence of numerical values given by*

$$P_{T(2,3+q)} = A_q(\tau,\omega)P_{T(2,3)} + B_q(\tau,\omega)P_{T(2,2)} \quad (q \in \mathbb{N}),$$

where $A_q(\tau,\omega)$ *and* $B_q(\tau,\omega)$ *are known functions of* τ *and* ω*, with initial conditions* $P_{T(2,3)}$ *and* $P_{T(2,2)}$.

Sketch of proof.

Apply (P.2) to



to obtain

$$P_{T(2,n+2)} = a^{-1}zP_{T(2,n+1)} + a^{-2}P_{T(2,n)}.$$

Recursively, we have

$$P_{T(2,n+2)} - \alpha P_{T(2,n+1)} = \beta^{n-1}\left[P_{T(2,3)} - \alpha P_{T(2,2)}\right], \quad n \geq 2,$$

and after some algebra

$$P_{T(2,n)} = \left(\frac{\beta^{n-2} - \alpha^{n-2}}{\beta - \alpha}\right) P_{T(2,3)} - \left(\alpha\beta\frac{\beta^{n-3} - \alpha^{n-3}}{\beta - \alpha}\right) P_{T(2,2)}, \quad n \geq 4\,.$$

Hence, by setting $k = e^{2\omega}$ ***and*** $a = e^{\tau}$***, we have:***

$$P_{T(2,3+q)} = A_q(\tau,\omega) P_{T(2,3)} + B_q(\tau,\omega) P_{T(2,2)} \quad (q \in \mathbb{N})\,,$$

with

$$A_q(\tau,\omega) = \frac{e^{2(1+q)\omega} - (-1)^{1+q} e^{-2(1+q)\omega}}{e^{q\tau}(e^{2\omega} + e^{-2\omega})}\,, \quad B_q(\tau,\omega) = \frac{e^{2q\omega} - (-1)^{q} e^{-2q\omega}}{e^{(1+q)\tau}(e^{2\omega} + e^{-2\omega})}\,.$$

and

$$P_{T(2,3)} = 2a^{-2} + a^{-2}z^2 - a^{-4}\,, \qquad P_{T(2,2)} = a^{-1}z + (a^{-1} - a^{-3})z^{-1}.$$

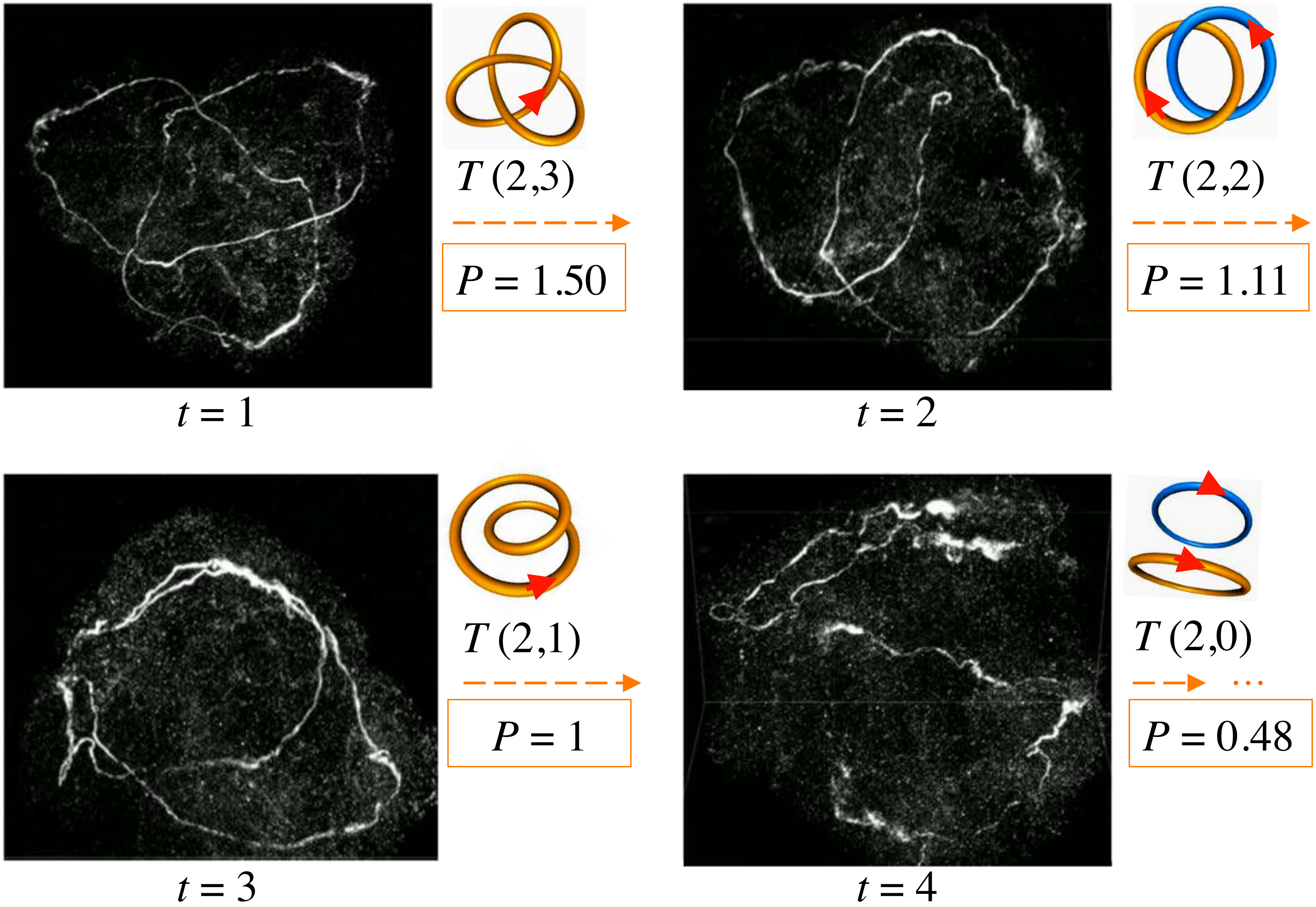
Since for mirror knot $P(K) \to P(\tilde{K})$ ***by changing***

$$a \to a^{-1} \ (\tau \to -\tau)\,, \quad z \to -z \ (\omega \to -\omega)\,,$$

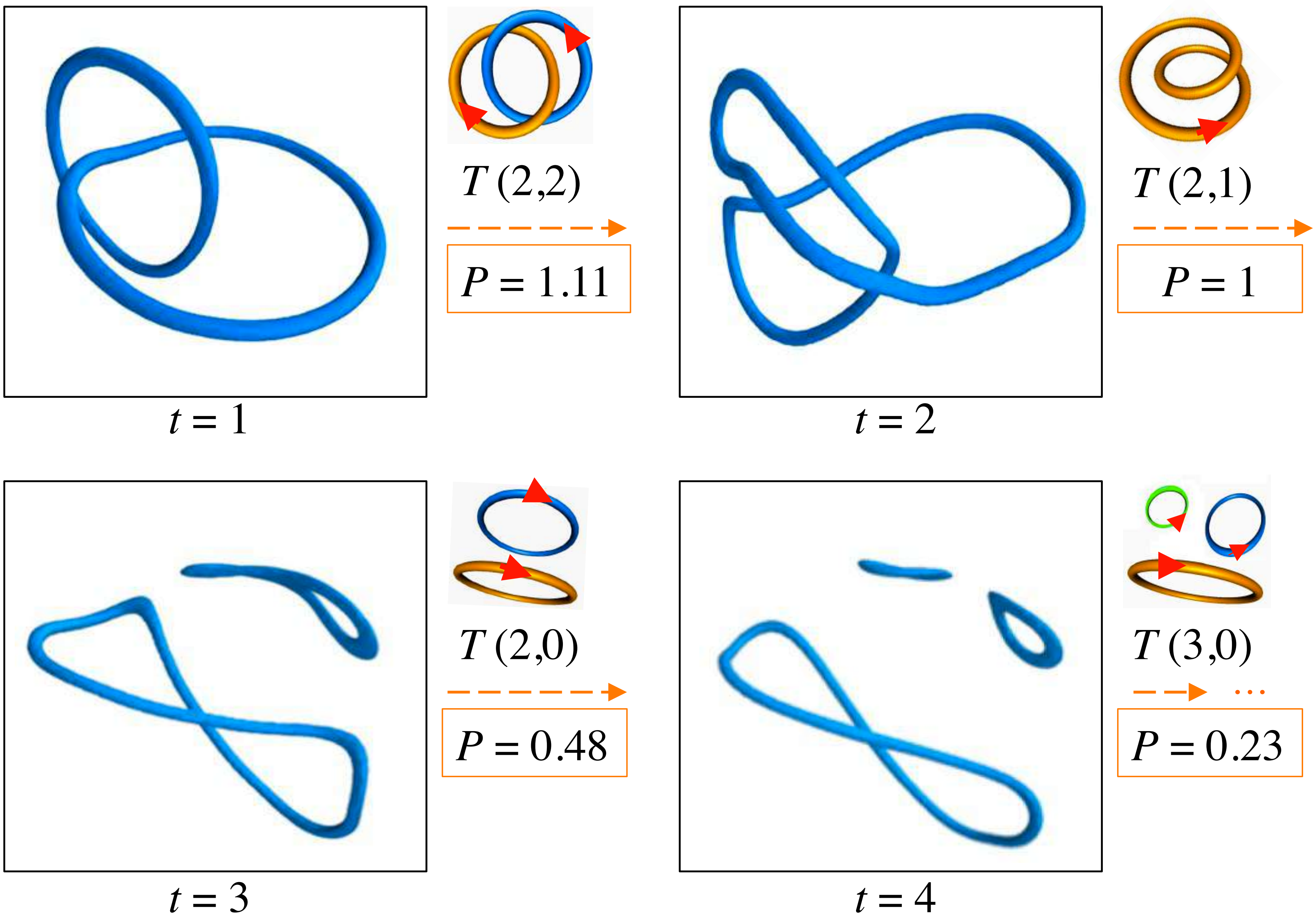
then

$$P\{T(2,n)\}_+ = P\{T(2,n)\}_- = P_{T(2,n)} \quad . \qquad \blacksquare$$

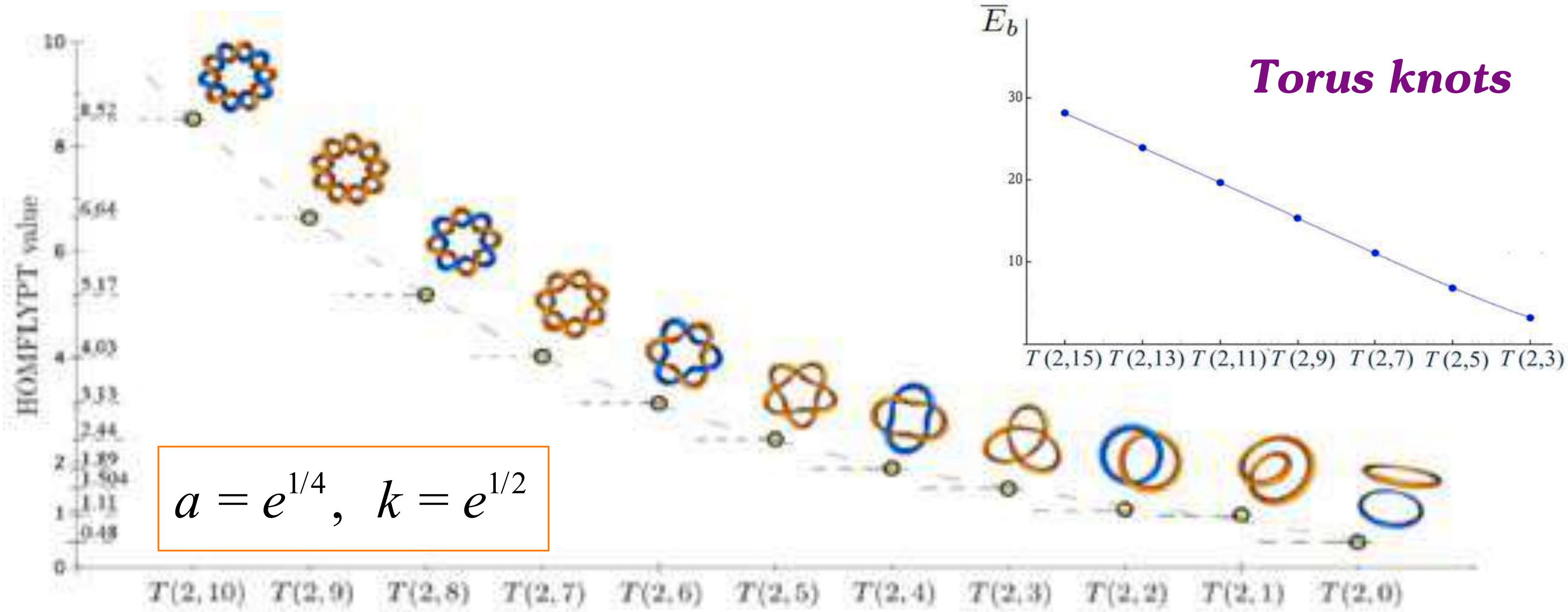
Vortex trefoil cascade process in water (Kleckner & Irvine 2013)



Vortex link cascade in BECs (Zuccher & Ricca 2017)



HOMFLYPT quantifies topological complexity



$$a = e^{1/4}, \quad k = e^{1/2}$$

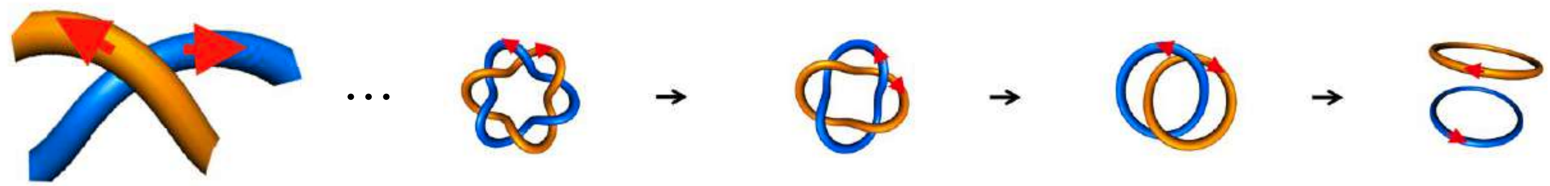
- ***Jones polynomial:*** $a = k^{-2} \quad (a = e, \ \ k = e^{-1/2})$;

$$V_{T(2,n)} = \frac{e^{-\frac{3}{2}n+4} + (-1)^{n-1} e^{-\frac{1}{2}n+2}}{e^{\frac{1}{2}} + e^{-\frac{1}{2}}} V_{T(2,3)} + \frac{e^{-\frac{1}{2}(3n-7)} + (-1)^{n-2} e^{-\frac{1}{2}(n-1)}}{e^{\frac{1}{2}} + e^{-\frac{1}{2}}} V_{T(2,2)} \ .$$

- ***Alexander-Conway polynomial:*** $a = 1 \quad (a = 1, \ \ k = e^{-1/2})$;

$$\Delta_{T(2,n)} = \frac{e^{-\frac{n-2}{2}} + (-1)^{n-1} e^{\frac{n-2}{2}}}{e^{\frac{1}{2}} + e^{-\frac{1}{2}}} \Delta_{T(2,3)} + \frac{e^{-\frac{n-3}{2}} + (-1)^{n-2} e^{\frac{n-3}{2}}}{e^{\frac{1}{2}} + e^{-\frac{1}{2}}} \Delta_{T(2,2)} \ .$$

Cascade of oppositely oriented components (negative crossings)

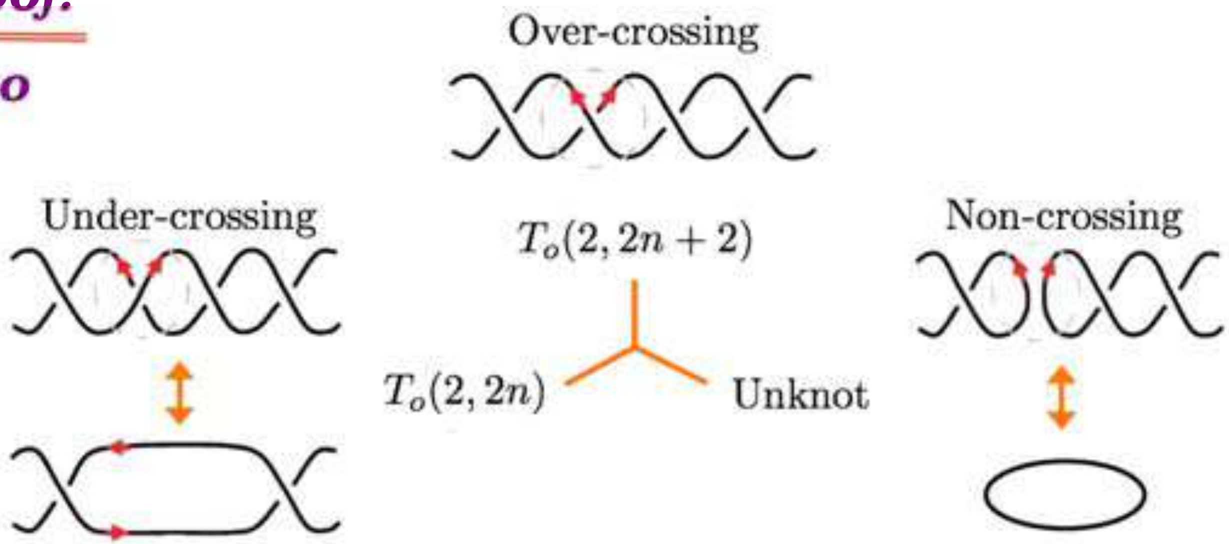
 : $\{T_o(2,n)\}$

- ***Lemma* (*Ricca & Liu 2017*). *Let us consider the ordered set of oppositely oriented torus links* $\{T_o(2,2n)\}$ (*n integer, $n \geq 1$*). *The HOMFLYPT polynomial* $P_{T_o(2,2n)}$ *is given by***

$$P_{T_o(2,2n)} = \frac{a^2-1}{az}\,a^{2n} + \frac{1-a^{2n}}{a^2-1}\,az\ .$$

Sketch of proof.

Apply (P.2) to



to obtain

$$aP_{T_o(2,2n)} - a^{-1}P_{T_o(2,2n+2)} = z \ ,$$

that is

$$P_{T_o(2,2n+2)} = a^2 P_{T_o(2,2n)} - az \ .$$

By applying the same relation recursively, we have

$$a^2 P_{T_o(2,2n)} = a^4 P_{T_o(2,2(n-1))} - a^3 z \ ,$$

$$a^4 P_{T_o(2,2(n-1))} = a^6 P_{T_o(2,2(n-2))} - a^5 z \ ,$$

$$\vdots$$

$$a^{2(n-1)+2} P_{T_o(2,2)} = a^{2(n-1)+4} P_{T_o(2,0)} - a^{2(n-1)+3} z \ ,$$

and by recursive substitution, we obtain

$$\begin{aligned} P_{T_o(2,2n+2)} &= a^{2(n-1)+4} P_{T_o(2,0)} - az(1 + a^2 + a^4 + \cdots + a^{2n}) \\ &= a^{2n+2} P_{T_o(2,0)} - az \frac{1 - a^{2(n+1)}}{1 - a^2} \qquad (n \geq 1) \ . \end{aligned}$$

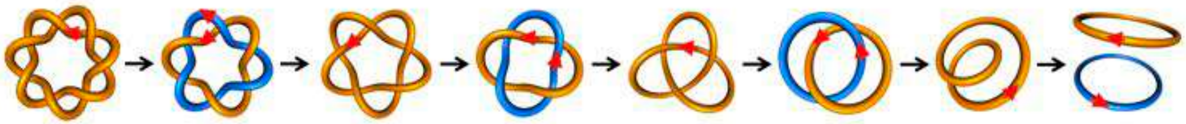
Since $P_{T_o(2,0)}$ *is the polynomial of the disjoint union of two unlinked unknots, given by*

$$P_{T_o(2,0)} = \frac{a - a^{-1}}{z} = \delta \ ,$$

we have the statement:

$$P_{T_o(2,2n)} = \frac{a^2 - 1}{az}\, a^{2n} + \frac{1 - a^{2n}}{a^2 - 1}\, az \ . \qquad \blacksquare$$

Table of numerical values: comparative analysis

$\{T(2,n)\}$: $\cdots$ 

Numerical values for torus knots and *co-oriented* torus links ($Wr = Tw = 1/2$)											
	$T(2,10)$	$T(2,9)$	$T(2,8)$	$T(2,7)$	$T(2,6)$	$T(2,5)$	$T(2,4)$	$T(2,3)$	$T(2,2)$	$T(2,1)$	$T(2,0)$
HOMFLYPT: $a = e^{1/4}, k = e^{1/2}$	8.52	6.64	5.17	4.03	3.13	2.44	1.89	1.50	1.11	1	0.48
Jones: $\tau = e^{-1}$	−0.01	0.02	−0.03	0.05	−0.09	0.15	−0.25	0.40	−0.69	1	−2.26
Alexander-Conway: $t = e^{-1}$	−65.81	39.92	−24.20	14.70	−8.88	5.44	−3.22	2.08	−1.04	1	-

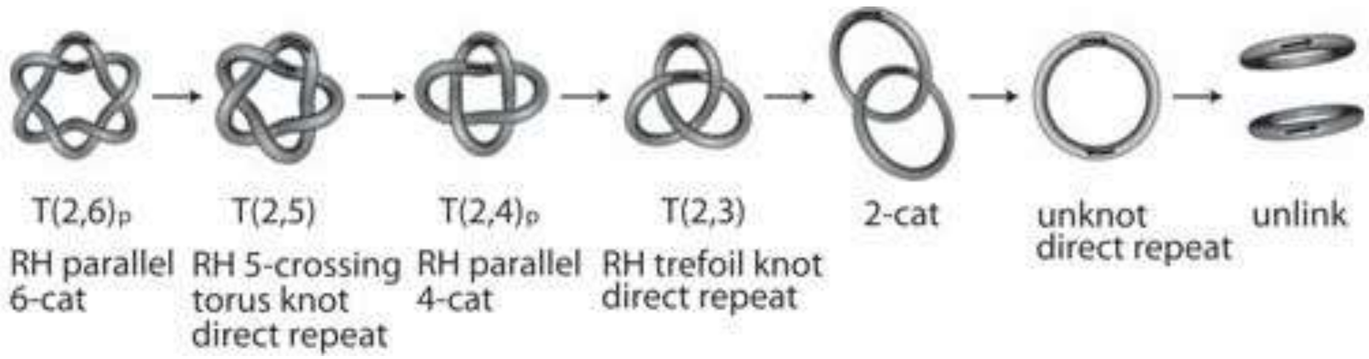
$\{T_o(2,n)\}$: $\cdots$ 

Numerical values for *oppositely oriented* torus links ($Wr = Tw = -1/2$)											
	$T_o(2,10)$	-	$T_o(2,8)$	-	$T_o(2,6)$	-	$T_o(2,4)$	-	$T_o(2,2)$	-	$T_o(2,0)$
HOMFLYPT: $a = e^{-1/4}, k = e^{-1/2}$	1.93		1.85		1.71		1.48		1.11		0.48
Jones: $\tau = e$	−0.44		−0.45		−0.45		−0.48		−0.69		−2.25
Alexander-Conway: $t = e^{1/2}$	0		0		0		0		0		-

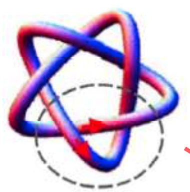
Conclusions and outlook

- ***Adapted HOMFLYPT is the best quantifier of cascade processes:***
  - P_K ***provides monotonic behavior consistently;***
  - ***numerical values more robust and reliable markers for diagnostics;***
  - $P_{T(2,2n)}/c_{\min} \approx 0.5$ ***,*** $(0 \leq n \leq 6)$ ***(except for the unknot).***

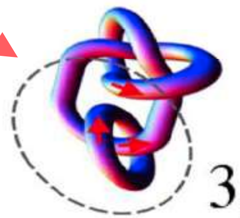
- ***Same cascade in recombinant DNA plasmids (Shimokawa et al. 2013):***



- ***Optimal path to cascade?***



$T(2,5)$



$3_1\#2_{2.1}$

$$P_{T(2,6)} = 3.13 \rightarrow P_{T(2,5)} = 2.44 \rightarrow P_{T(2,4)} = 1.89 \rightarrow P_{T(2,3)} = 1.50$$

$$P_{T(2,6)} = 3.13 \rightarrow P_{3_1\#3_1} = 2.26 \rightarrow P_{3_1\#2_{2.1}} = 1.66 \rightarrow P_{T(2,3)} = 1.50$$

These are the

Ravello Lecture Notes

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The School was held under the auspices of the Istituto Nazionale di Alta Matematica (INdAM)

All the material can be directly downloaded from the lecturer's personal website at:

http://www.matapp.unimib.it/~ricca/post-doctoral.html