Lecture 4

- ***Topological transition of soap films:***

  - *Seifert surfaces and soap films*
  - *The Plateau problem: the catenoid*
  - *Topological transition from 1-sided to 2-sided surface*
  - *Local analysis of the twisted saddle catastrophe*
  - *Geometry and energy considerations during transition*

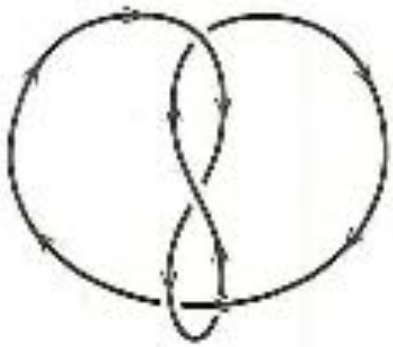
- ***Selected references***

Goldstein, RE, Moffatt, HK, Pesci, AI & Ricca, RL 2010 A soap film Möbius strip changes topology with a twist singularity. PNAS 107, 21979.

Maggioni, F & Ricca, RL 2006 Writhing and coiling of closed filaments. Proc R Soc A 462, 3151.

Ricca, RL 2009 Structural complexity and dynamical systems. In Lectures on Topological Fluid Mechanics (ed. R.L. Ricca), pp. 169-188. Springer-CIME Lecture Notes in Mathematics 1973. Springer-Verlag.

Creating a Seifert surface (orientable spanning surface)

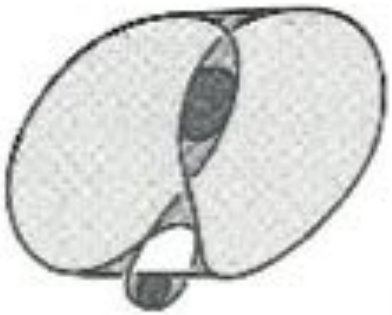


1.

2.

3.

4.



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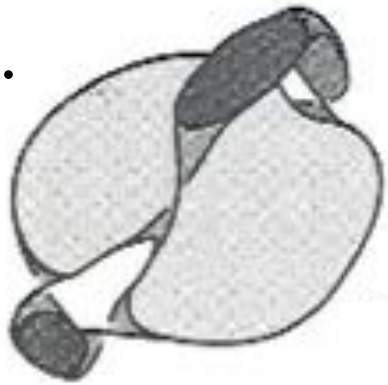
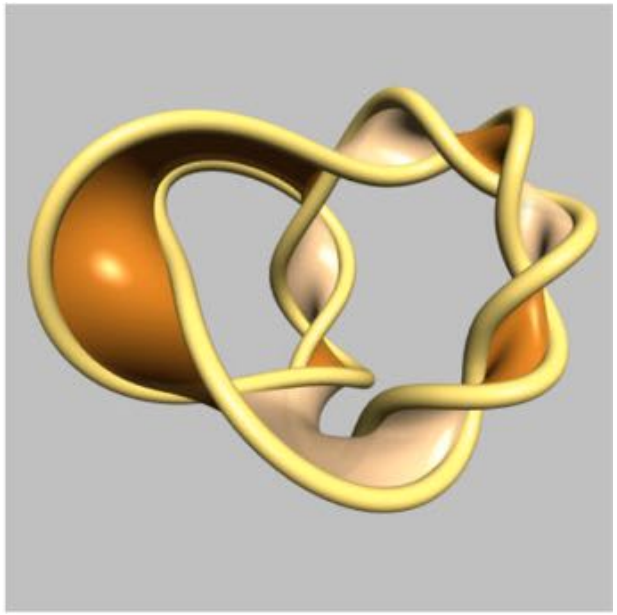
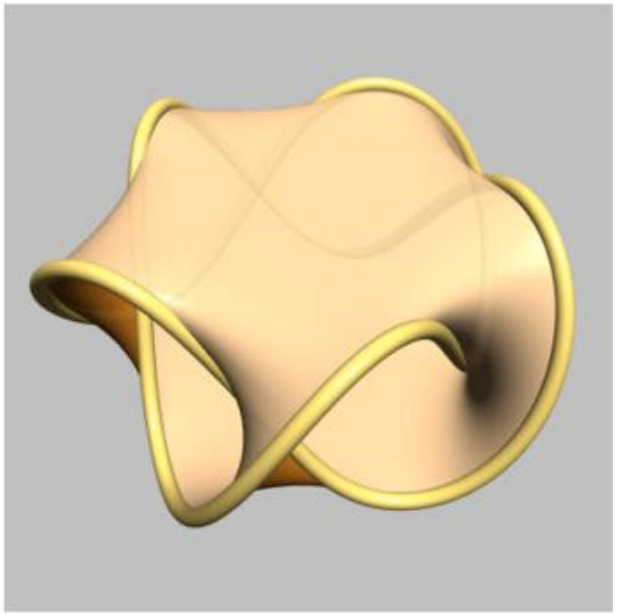


Examples of Seifert surfaces (Van Vijk, 2009)

a

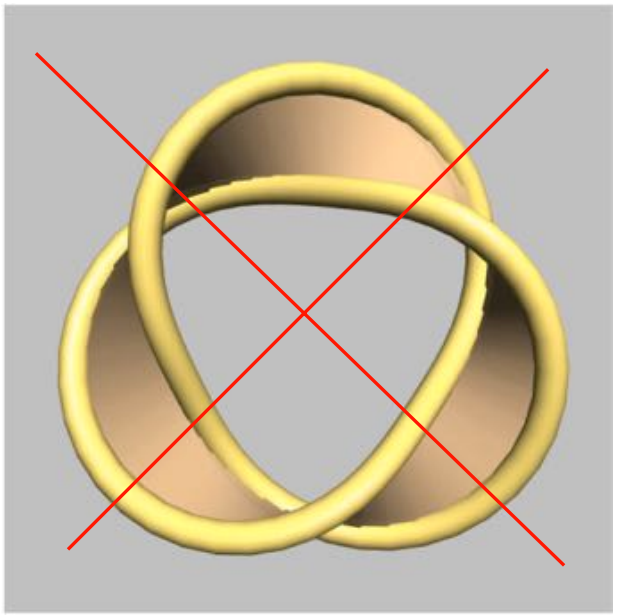


b



c

d



non-orientable

Gauge fixing and Seifert surface

- ***Vector potential***

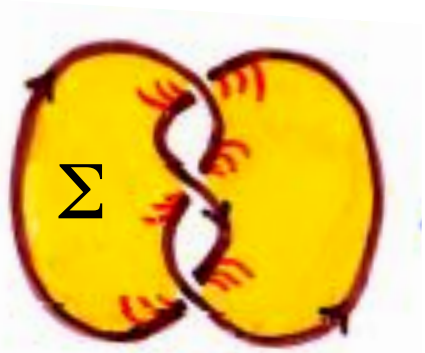
  Let Σ ***be a closed, orientable surface in*** $\mathbb{R}^3$***; then***

$$\int_\Sigma \mathbf{B}\cdot\hat{\nu}\,\mathrm{d}^2\mathbf{X} = 0 \quad \Rightarrow \quad \nabla\cdot\mathbf{B} = 0 \quad .$$

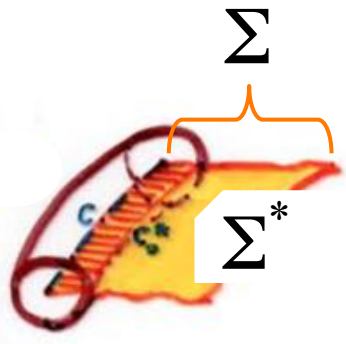
  Hence, there exists a vector potential $\mathbf{A}$***, single-valued, such that***

$$\mathbf{B} = \nabla\times\mathbf{A} \;\;,\; \nabla\cdot\mathbf{A} = 0 \qquad \textit{with} \quad \mathbf{A} = O\left(|\mathbf{X}|^{-3}\right) \quad \textit{as} \quad |\mathbf{X}|\to\infty \quad .$$

- ***Seifert surface:***



orientable

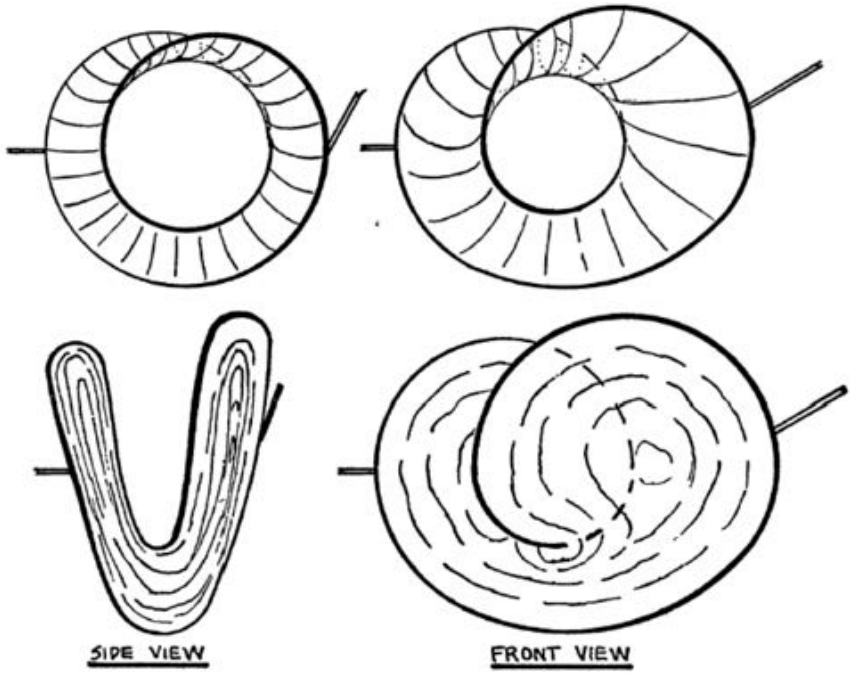


- ***Gauge invariance of helicity: take*** $Lk(C,C^*) = 0$ ***, where*** $C^* = \Sigma\cap\partial\mathcal{T}$***;***

  then: $\int_{\Sigma^*}\mathbf{B}\cdot\hat{\nu}\,\mathrm{d}^2\mathbf{X} = \oint_{C^*}\mathbf{A}\cdot\mathrm{d}\mathbf{X} = 0$ ***,*** ***where*** $C^* = \partial\Sigma^*$ ***.***

A distinguished precedent: Richard Courant (1888-1972)

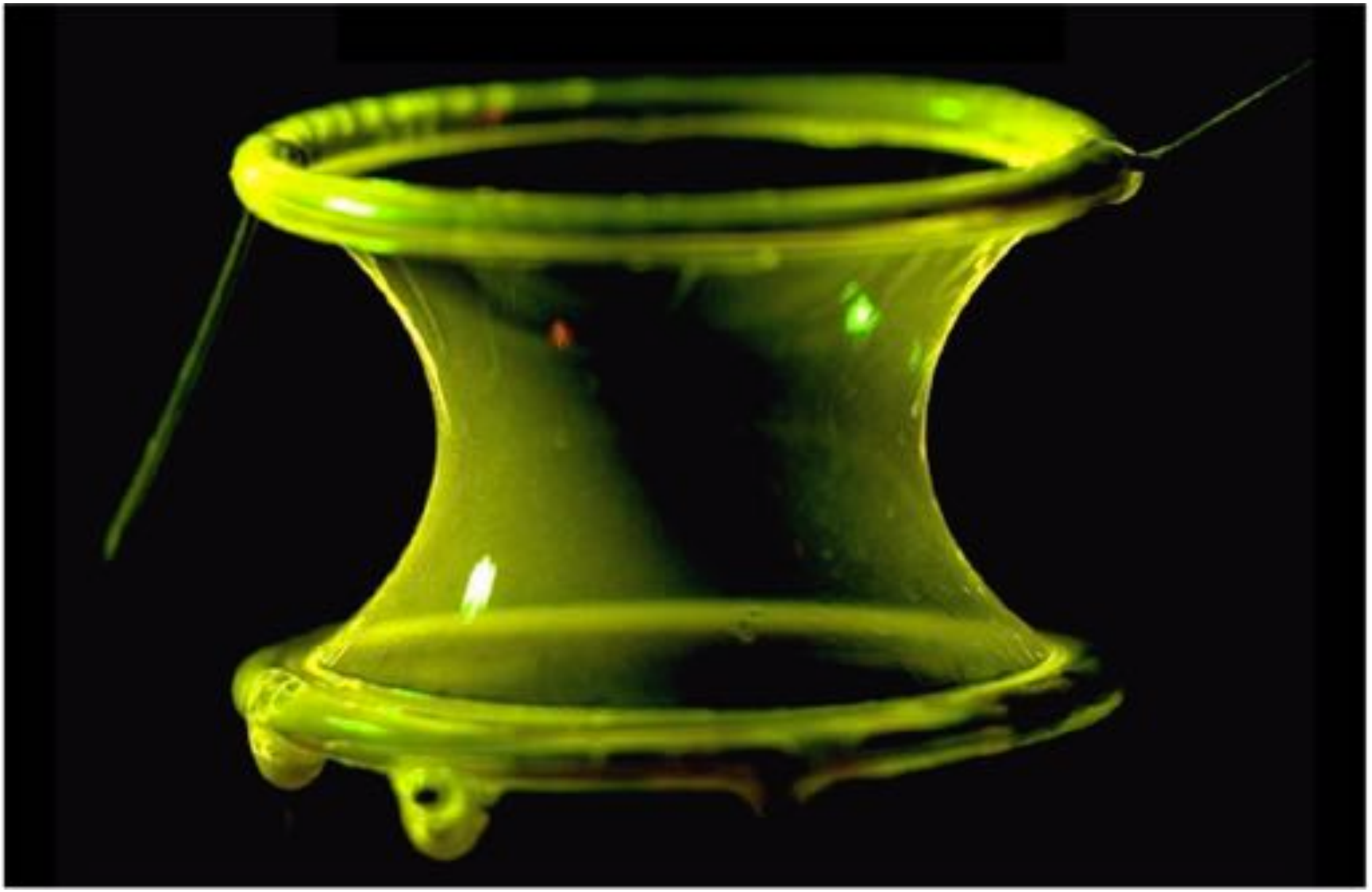
"… a one-sided soap-film in the form of a Möbius strip is easily formed, and [that] it can jump to a two-sided film if the wire boundary is suitably distorted …"



***Courant 1940** Soap film experiments with minimal surfaces. Amer. Math. Mon. 47, 167–174.*

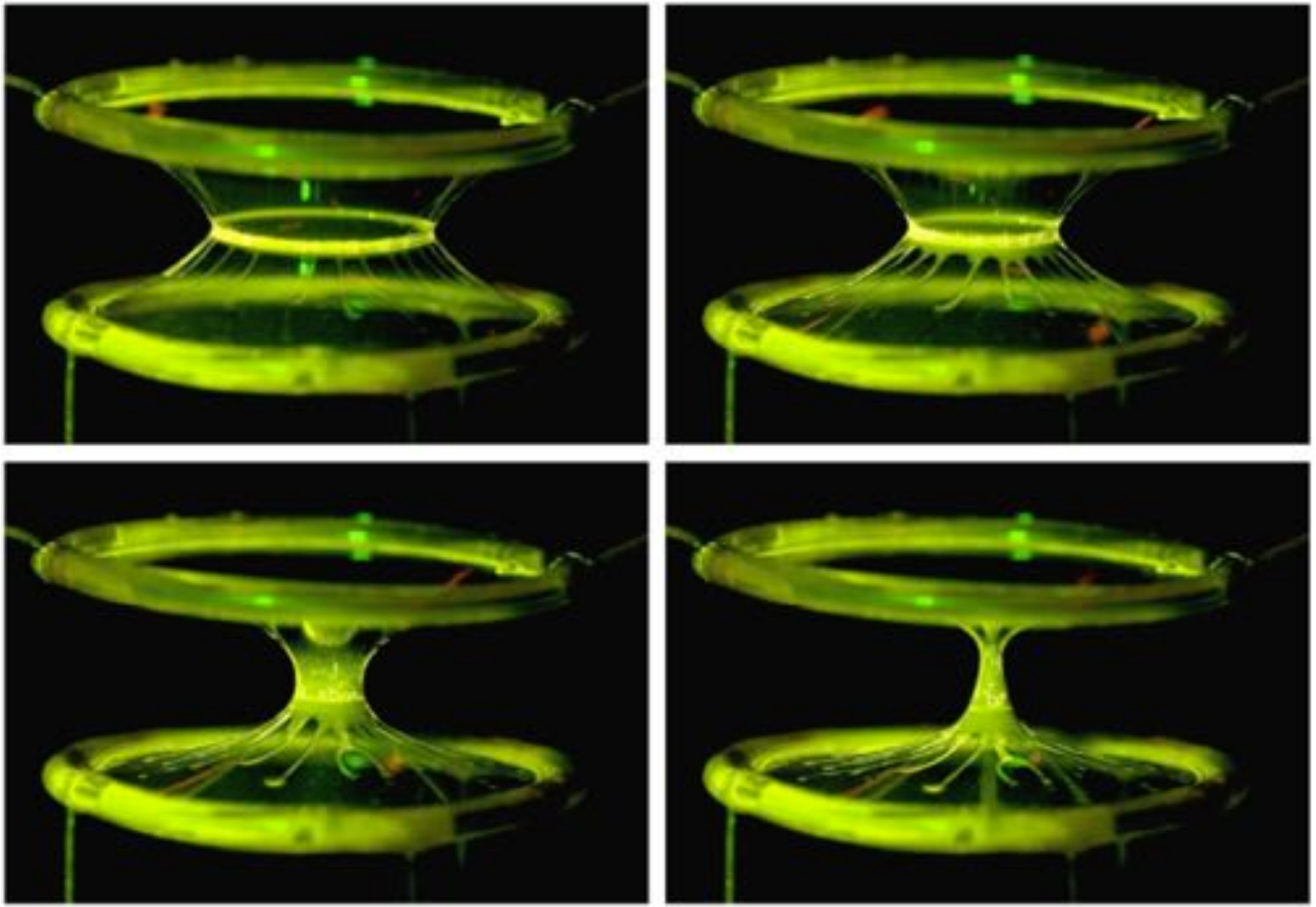
Collapse of the catenoid (singularity in the bulk)

Earlier studies: Euler 1744; Lagrange 1762



(Goldstein's Lab, DAMTP, Cambridge 2010)

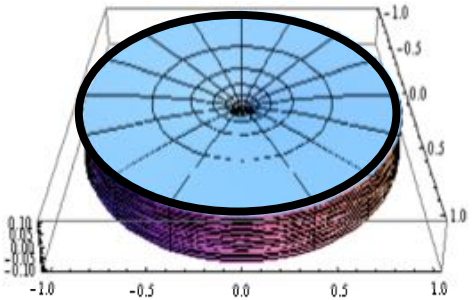
Search for minimum area solution of the Plateau problem



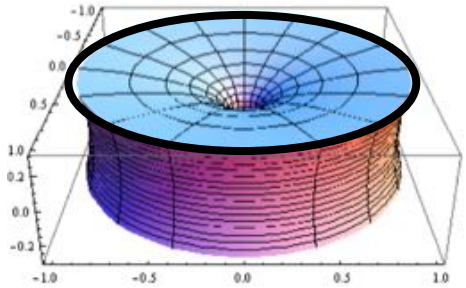
(Goldstein's Lab, DAMTP, Cambridge 2010)

Search for minimum area solution of the Plateau problem

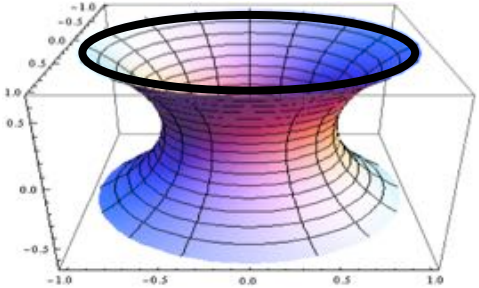
Recent works: Robinson &Steen 2001; Nitsche & Steen 2004



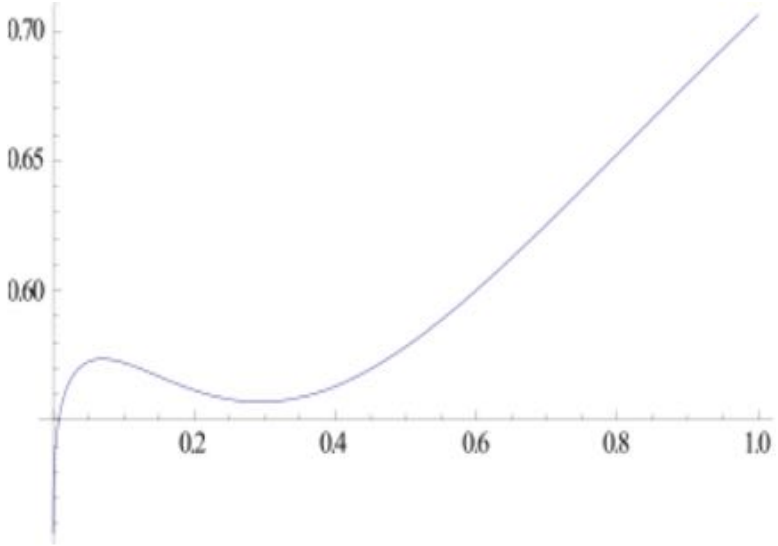
(a) b = 0.1



(b) b = 0.3

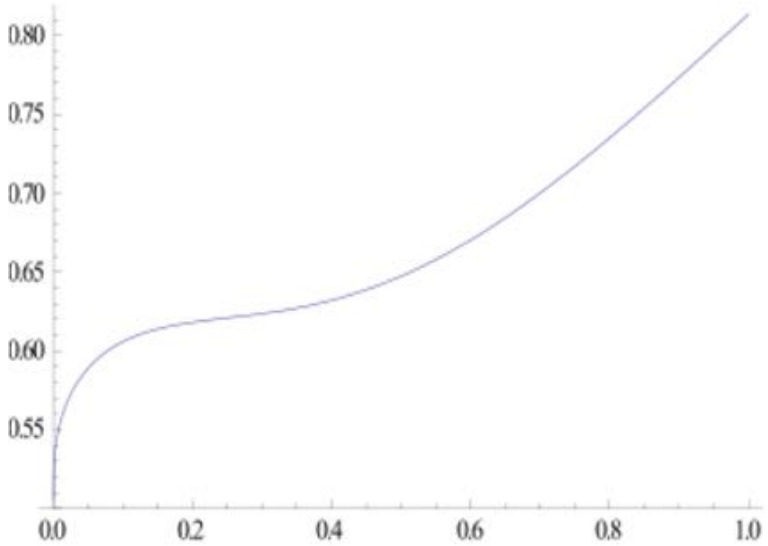


(c) $b = b_c = 0.6627$



b = 0.6

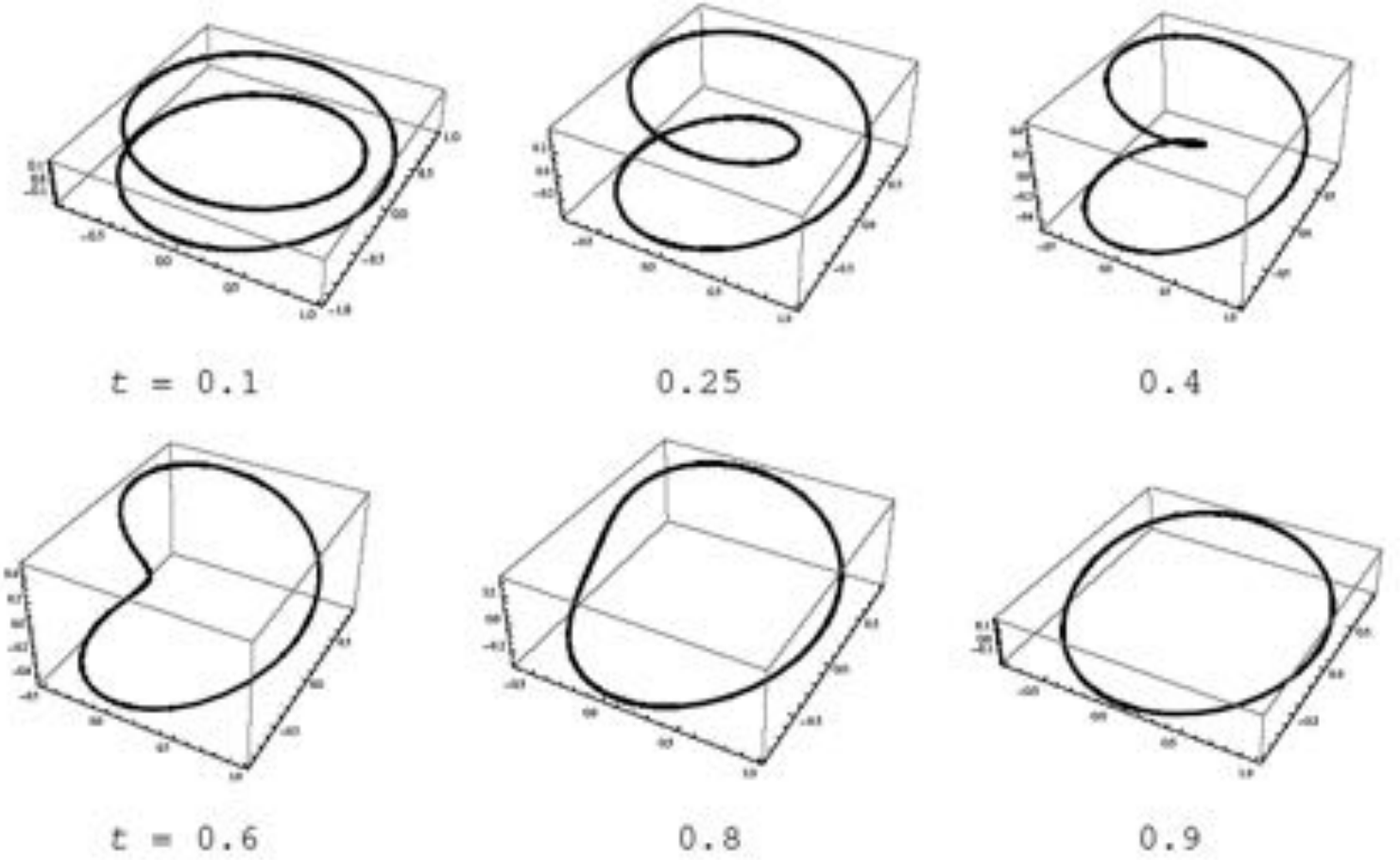
2 solutions, 1 stable, 1 unstable



b = 0.7

no minimum area solution

Unfolding of a space curve



$$C:\quad \mathbf{X}(s;t)=\begin{cases} x(s;t)=\left[-t\cos s+(1-t)\cos 2s\right]/l(t) \\ y(s;t)=\left[-t\sin s+(1-t)\sin 2s\right]/l(t) \\ z(s;t)=\left[-2t(1-t)\sin s\right]/l(t) \end{cases}$$

$$0\le s\le 2\pi$$
$$0<t<1$$

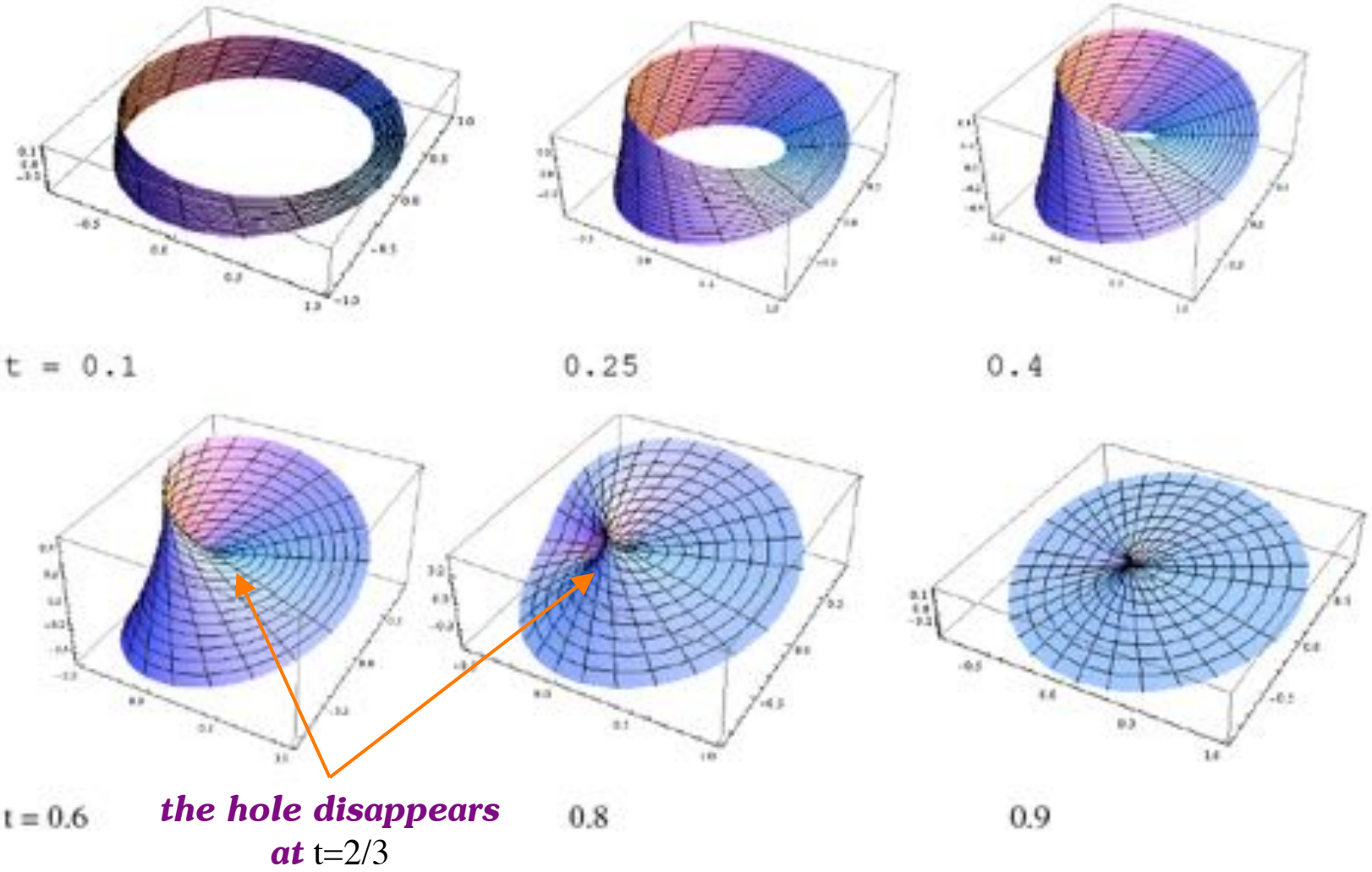
(Maggioni & Ricca 2006)

Modeling Möbius strip evolution by ruled surface

- ***From space curve to ruled surface:*** $\mathbf{X}(s;t) \to \mathbf{X}(s,\mu;\ t)\ ,\quad \mu \in [-1,1]$

$$\mathbf{X}(s,\mu;\ t) = \left[\mu t\cos s + (1-t)\cos 2s,\ \mu t\sin s + (1-t)\sin 2s,\ 2\mu t(1-t)\sin s\right]$$

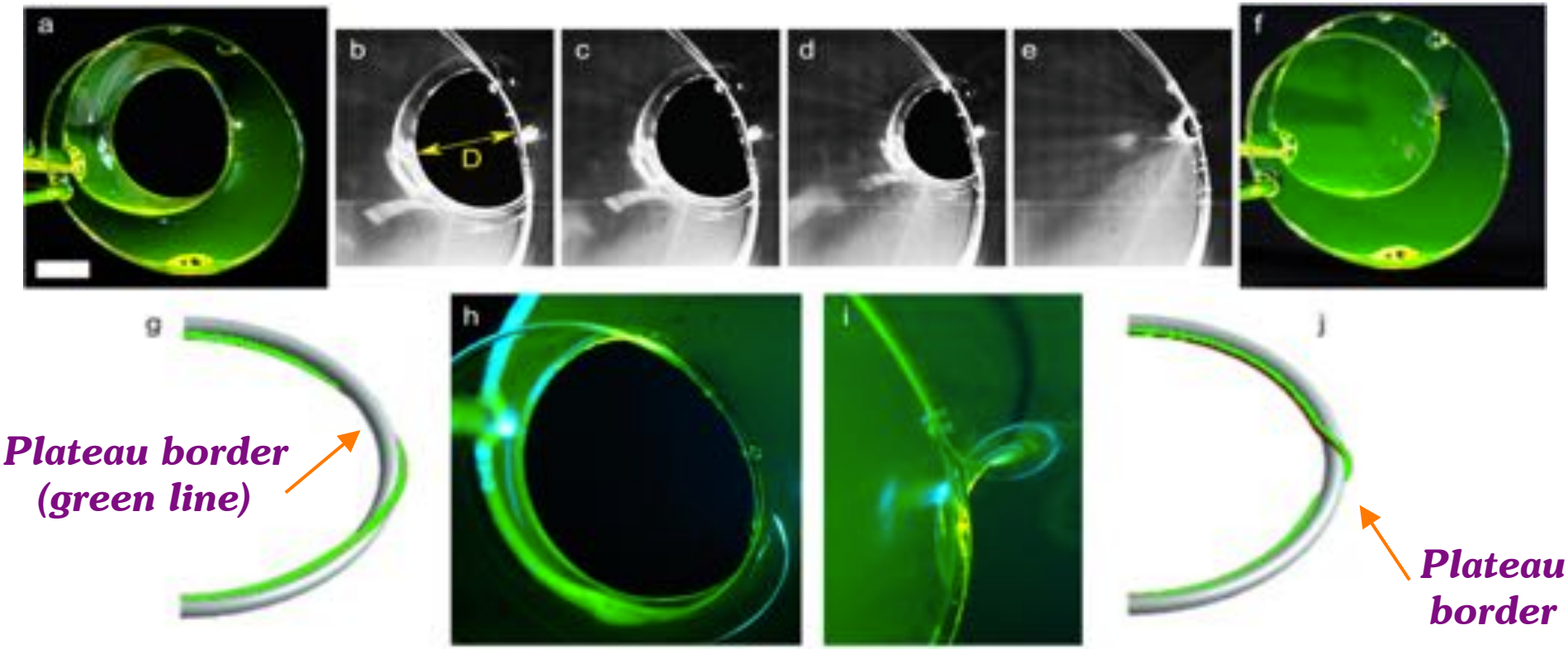
$\mu = 1$ ***: the strip reduces to a circle of radius*** 1–t ***in the plane*** $z = 0$***.***



Evolution of the soap film surface (Goldstein et al. 2010)

- ***Poisson-Laplace-Young equation:***

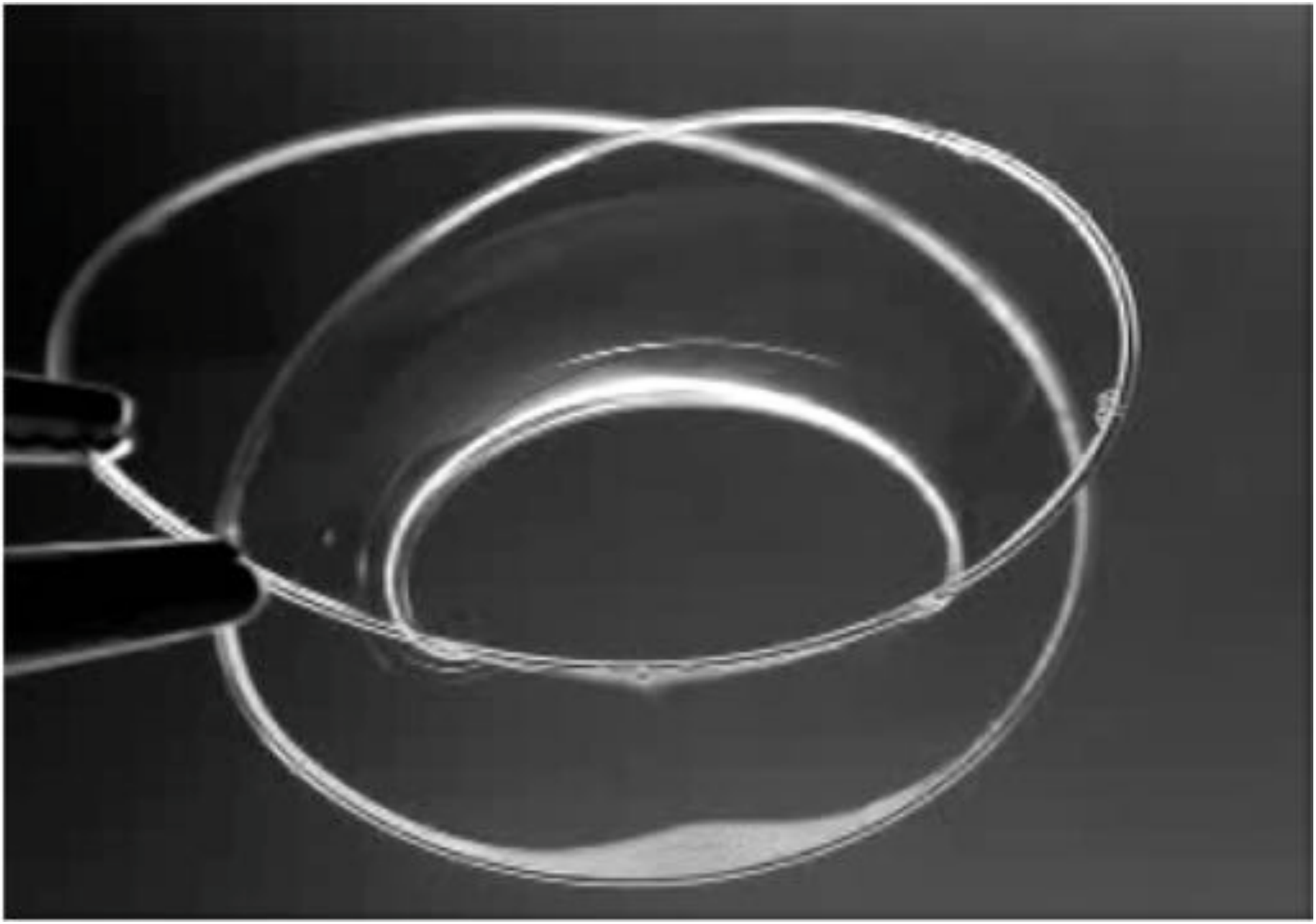
$$p = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = 2\sigma H \quad \Rightarrow \quad p = 0 \;\Rightarrow\; H = 0$$



before collapse ***after collapse***

Frames from high-speed movie at intervals of 5.4 ms .

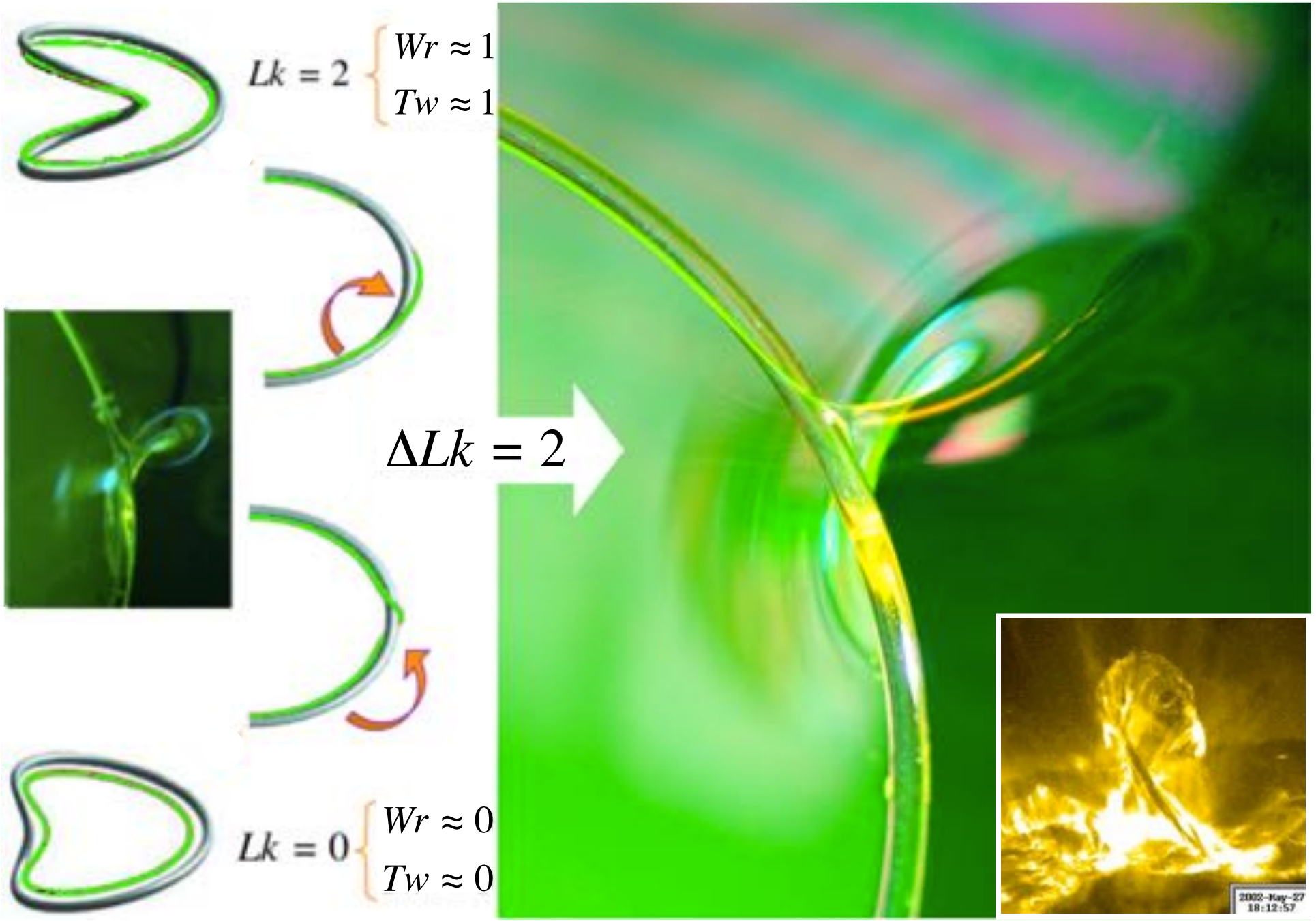
Topology change with a twist singularity (Goldstein et al. 2010)



Expanded view of the Plateau border near wire frame after collapse

(Goldstein's Lab, DAMTP, Cambridge 2010)

Expanded view of the twisted fold singularity after collapse



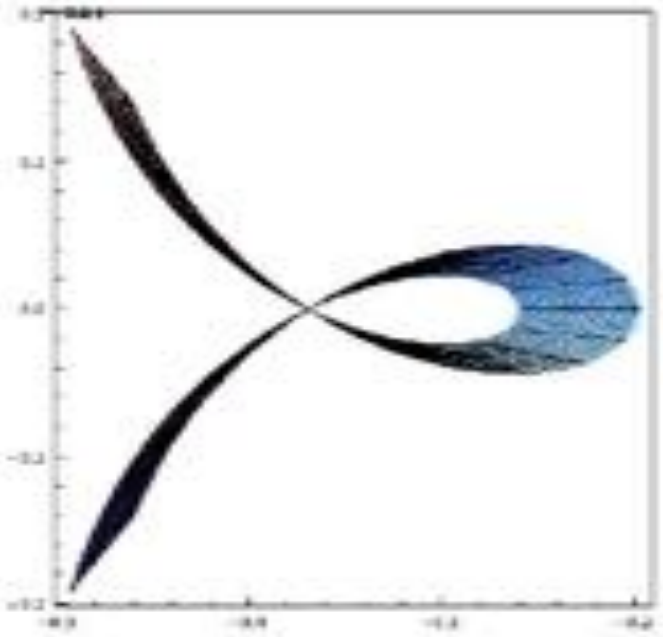
Ruled surface: topological transformation through a twisted cubic

- ***Passage through an inflexion point:*** $s = \mu = 0,\ t = t_c = 2/3:\quad c = 0$

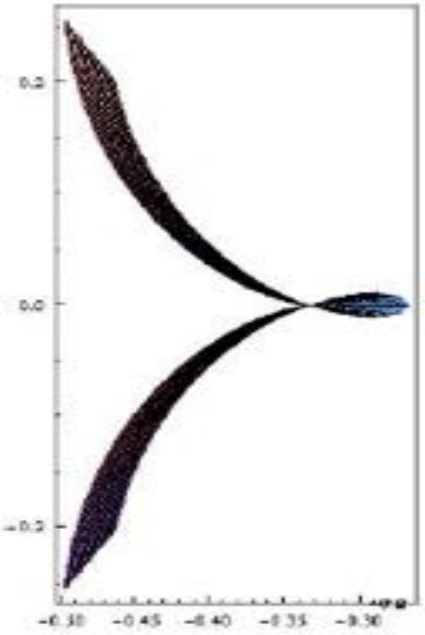
near the inflexion point:

$$\begin{cases} x_c = -1/3 - 1/3 s^2 \\ y_c = -1/3 s^3 \\ z_c = -4/9 s \end{cases}$$

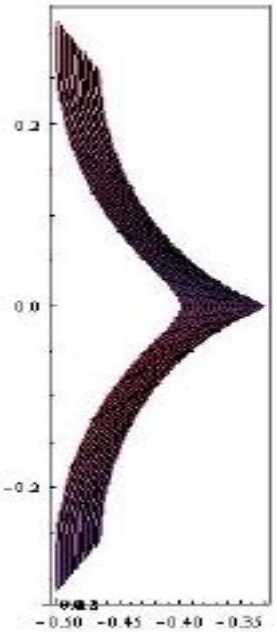
z-plane projection of a portion of the surface: $s \in [-1,1]\ ,\ \mu \in [-1,-0.9]\ .$



$t = 0.63$



$t_c = 2/3$



$t = 0.7$

Reidemeister type I move (Moffatt & Ricca 1992)

Ruled surface: change of topology by a cusp catastrophe

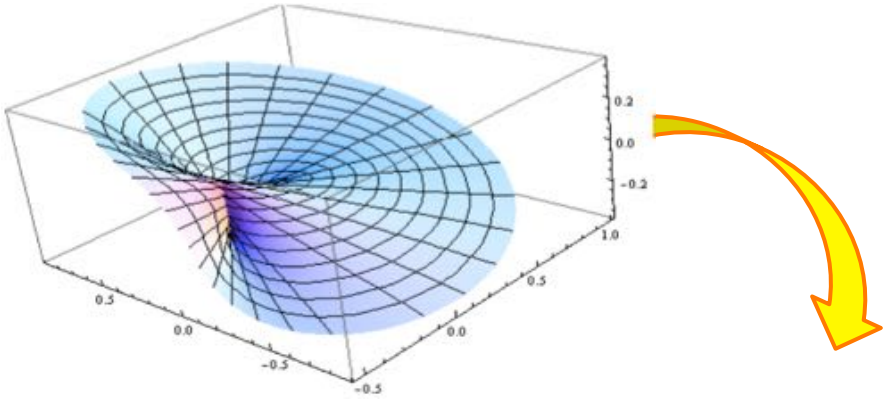
- ***Emergence of a twisted fold: local analysis***

origin on boundary

$$\begin{cases} s = 0 \\ \mu = -2(1-t)/t \end{cases} \Rightarrow \begin{matrix} 2/3 \le t < 1 \\ -1 \le \mu < 0 \end{matrix} \Rightarrow \begin{matrix} t_c = 2/3 \\ \mu = -1 \end{matrix}$$

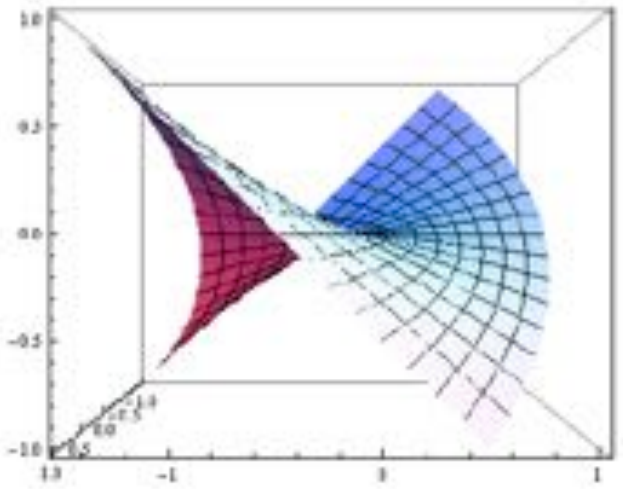
- ***Algebraic local form near*** $s = 0, \quad \varepsilon = \mu + 2(1-t)/t = 0$ ***:***

$$\begin{cases} X = 2(1-t)\left[t\varepsilon - 2(1-t)s^2\right] \\ Y = 4(1-t)^2 ts\varepsilon \\ Z = -2(1-t)s \end{cases}$$



universal twisted saddle geometry

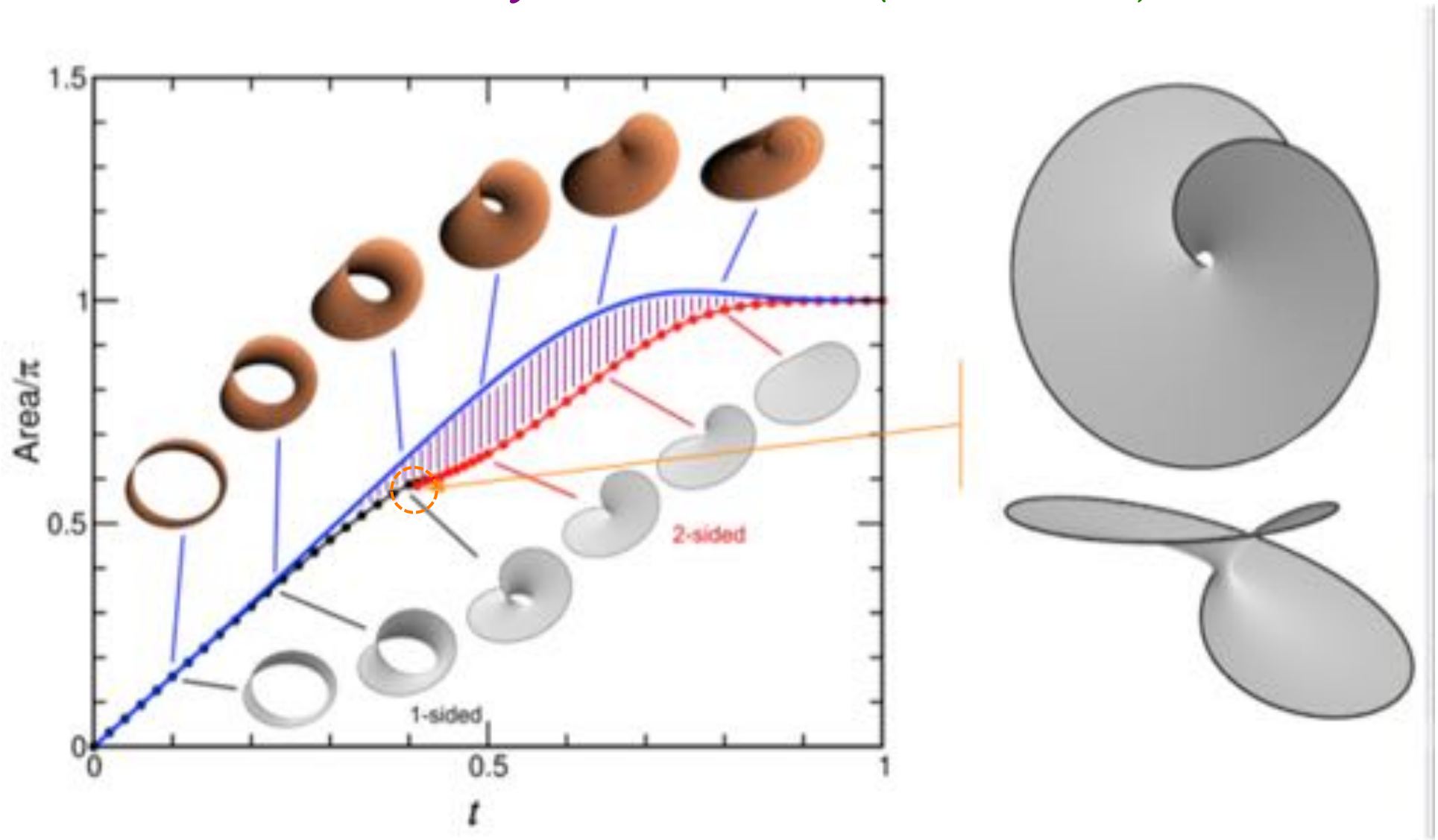
$$Y = -\frac{1}{2}Z^3 - XZ$$



Conjecture (for the soap film): if A_1 ***= area of 1-sided surface,*** A_2 ***= area of 2-sided, Then*** $A_1 < A_2$ ***for*** $t < 2/3$ ***and*** $A_1 > A_2$ ***for*** $t > 2/3$.

Energy and geometry from 1-sided to 2-sided surface

- ***Numerical evolution by* Surface Evolver *(Brakke 1992)***



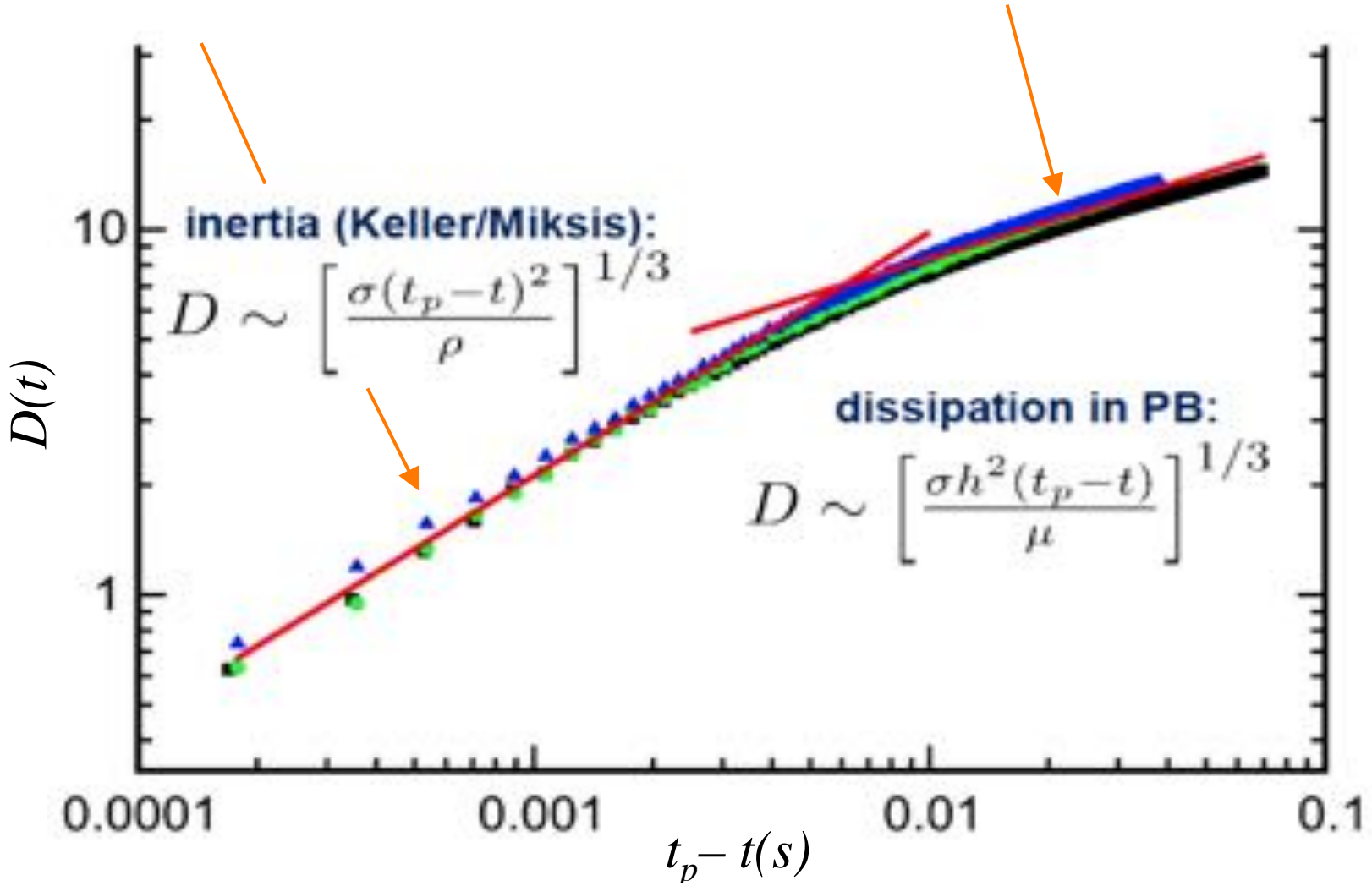
***ruled surface:* Mathematica** **minimal surface: Surface Evolver**

Dynamics of the topological transition

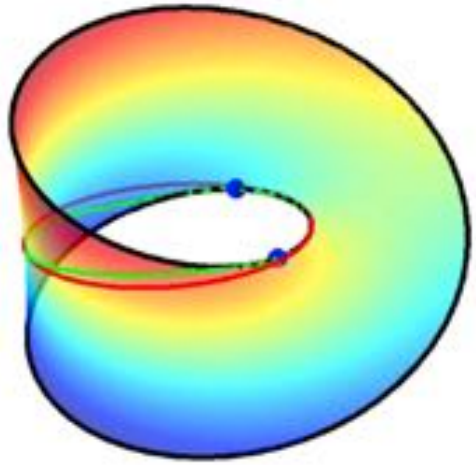
Throat diameter $D(t)$, averaged over multiple runs; t_p= pinch time

2/3 ***exp. closer to pinch time under air inertia control***

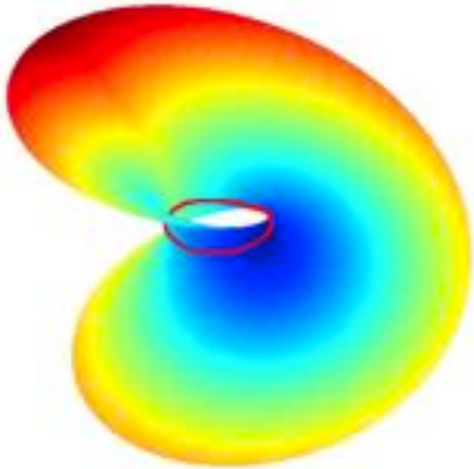
1/3 ***exp far from the singularity under viscous control***



Geometric analysis of surface dynamics: location of systolic curves



(a)



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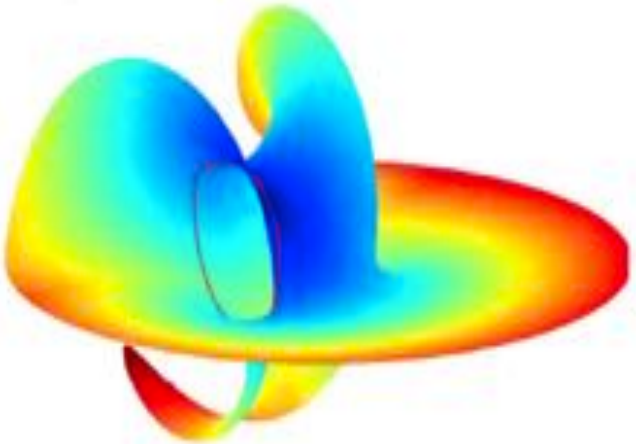

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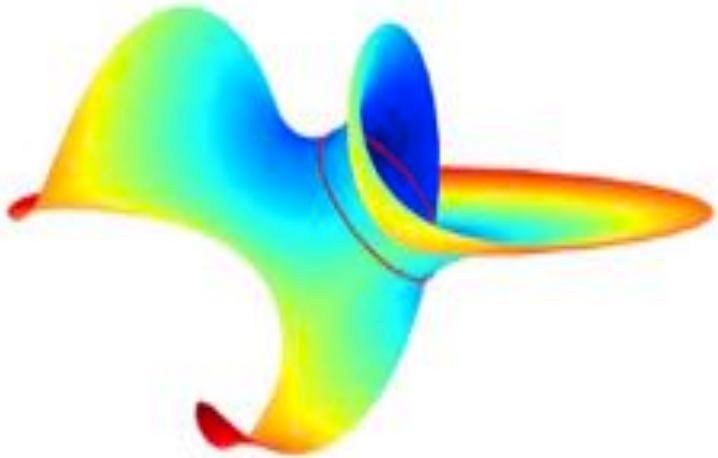

(d)

(Goldstein et al. 2014)