## *Lecture 3*

- ***Magnetic knots and groundstate energy spectrum:***

  *- Magnetic relaxation*

  *- Topology bounds the energy*

  *- Inflexional instability of magnetic knots*

  *- Constrained minimization of magnetic energy*

  *- Groundstate energy spectra of magnetic knots and links*

  *- Bending energy spectra: Magnetic vs. elastic systems*

- ***Selected references***


Maggioni, F & Ricca, RL 2009 On the groundstate energy of tight knots. *Proc R Soc A* 465, 2761.

Ricca, RL 2008 Topology bounds energy of knots and links. *Proc R Soc A* 464, 293.

Ricca, RL & Maggioni, F 2014 On the groundstate energy spectrum of magnetic knots and links. *J Phys A: Math & Theor* 47, 205501.

## *Magnetic relaxation*

- ***Magnetic relaxation (Moffatt 1985): consider*** $\varphi : \mathcal{L}_{\mathrm{n}} \rightarrow \mathcal{L}_{\mathrm{n},\varphi}$ ***in a viscous and perfectly conducting fluid;***

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \int_{V(\boldsymbol{B})} \boldsymbol{B} \cdot \frac{\partial \boldsymbol{B}}{\partial t} \, \mathrm{d}^3\boldsymbol{X} = \int_{V(\boldsymbol{B})} \boldsymbol{B} \cdot \left[ \nabla \times \left( \boldsymbol{u} \times \boldsymbol{B} \right) \right] \mathrm{d}^3\boldsymbol{X}$$

$$= - \int_{V(\boldsymbol{B})} \boldsymbol{u} \cdot \left[ \left( \nabla \times \boldsymbol{B} \right) \times \boldsymbol{B} \right] \mathrm{d}^3\boldsymbol{X} = - \int_{V(\boldsymbol{B})} \boldsymbol{u} \cdot \left[ \boldsymbol{J} \times \boldsymbol{B} \right] \mathrm{d}^3\boldsymbol{X} \; .$$

***Near equilibrium we have:***

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \propto \boldsymbol{u}$$

***hence***

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p + \boldsymbol{J} \times \boldsymbol{B} \quad \Rightarrow \quad \boldsymbol{u} \propto -\nabla p + \boldsymbol{J} \times \boldsymbol{B} \; .$$

***Thus:***

$$\frac{\mathrm{d}M}{\mathrm{d}t} \propto - \int_{V(\mathcal{L}_{\mathrm{n}})} \left\| \boldsymbol{u} \right\|^2 \mathrm{d}^3\boldsymbol{X} \quad \Rightarrow \quad M\left( \mathcal{L}_{\mathrm{n},\varphi} \right) \downarrow_t$$

***with magnetic energy monotonically decreasing as*** $t \rightarrow \infty$ .

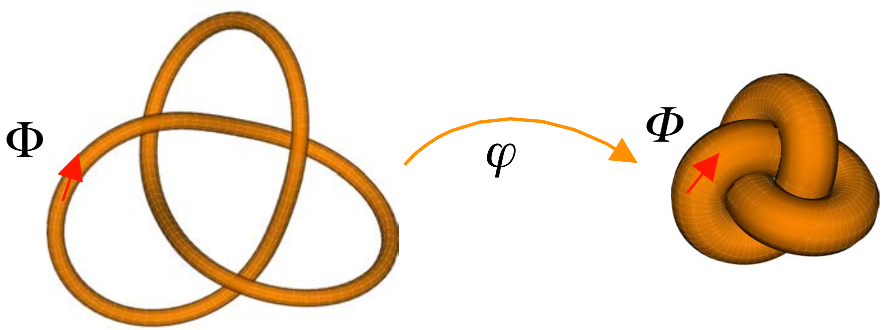
## *Magnetic relaxation to groundstate*

*Let* $\mathcal{L}_n$ *be a "zero-framed" magnetic link:* $\begin{cases} N \text{ components} & Lk_i = 0 \quad \forall i \\ \text{same flux } \Phi_i = \Phi \,. \end{cases}$

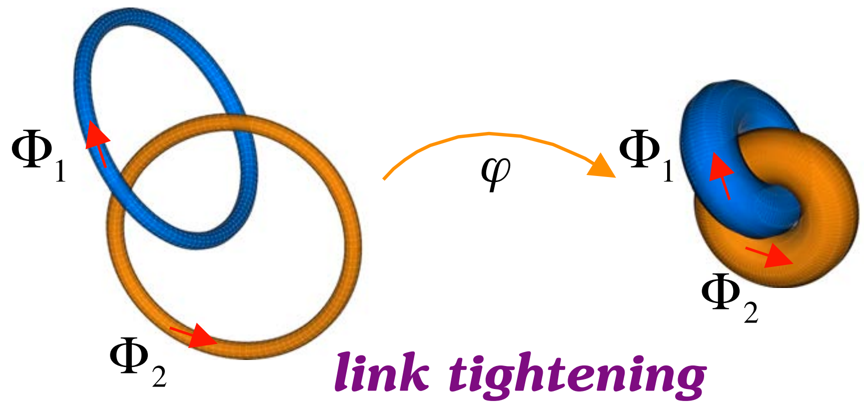
- ***Magnetic helicity:***

$$H = \int_{V(\mathbf{B})} \mathbf{A} \cdot \mathbf{B} \, \mathrm{d}^3\mathbf{x} = \Phi^2 \sum_{i \neq j} Lk_{ij} \quad .$$

- ***Magnetic relaxation (Moffatt 1985) under*** $\{V, \Phi_i\}$***-preserving flow:***




*knot tightening*




*link tightening*

- ***Theorem 1 (Freedman 1988; Moffatt 1990).***

*i)* $\exists\ {}^{+}\mathrm{l.b.}: \lim_{t \to \infty} M(t) = M_{\inf}$ *;* *ii)* $M_{\min} = m \dfrac{\Phi^2}{V^{1/3}}$ *,*

*where* $m$ *is a topological invariant of* $L_N$ *.*

## *Topology bounds the energy*

- **Theorem 2** *(Arnold 1974; Freedman & He 1991).*

$$\textit{i)}\quad M(t) \geq q\,|H|\ ; \qquad\qquad \textit{ii)}\quad M(t) \geq \left(\frac{2}{\pi}\right)^{1/3} \frac{\Phi^2 \overline{C}}{V^{1/3}}\ ,$$

*where* $q > 0$ *depends on the geometry of* $\operatorname{supp}(\mathbf{B})$ *.*

- **Theorem** *(Ricca 2008). Let* $\mathcal{L}_n$ *be a zero-framed link. Then, we have:*

$$\text{I)}\qquad q = \left(\frac{2}{\pi}\right)^{1/3} \frac{1}{V^{1/3}}\ : \qquad\qquad M(t) \geq \left(\frac{2}{\pi}\right)^{1/3} \frac{1}{V^{1/3}} |H|\ ,$$

$$\text{II)}\qquad m = \left(\frac{2}{\pi}\right)^{1/3} c_{\min}\ : \qquad M_{\min} = \left(\frac{2}{\pi}\right)^{1/3} c_{\min} \frac{\Phi^2}{V^{1/3}}\ .$$

*Proof:*

*From* $\Phi_i = \Phi$ *and* $Lk_i = 0 \quad (\forall i \in 1,\ldots,\mathrm{n})$ *we have*

$$H = \Phi^2 \left( \sum_{i=j} \overbrace{\cancel{Lk_i}}^{=\,\mathbf{0}} + \sum_{i \neq j} Lk_{ij} \right) = \Phi^2 \sum_{i \neq j} Lk_{ij}\ ;$$

*from* $\overline{C} = \sum_{i \neq j} \left\langle \sum_r |\varepsilon_r| \right\rangle$ *and* $\sum_{i \neq j} |Lk_{ij}| = \frac{1}{2} \sum_{i \neq j} \left| \sum_r \varepsilon_r \right|$ *, we have*

$$\bar{C} \geq \sum_{i \neq j} \left| \sum_r \varepsilon_r \right| = \sum_{i \neq j} \left| Lk_{ij} \right| \geq \left| \sum_{i \neq j} Lk_{ij} \right| .$$

*Therefore*

$$|H| = \Phi^2 \left| \sum_{i \neq j} Lk_{ij} \right| \leq \Phi^2 \bar{C} \qquad \text{or} \qquad \bar{C} \geq \frac{|H|}{\Phi^2} . \qquad (*)$$

*Now, from Theorem 2(ii), we have:*

$$M \geq \left( \frac{2}{\pi} \right)^{1/3} \frac{\Phi^2 \bar{C}}{V^{1/3}} \geq \left( \frac{2}{\pi} \right)^{1/3} \frac{\Phi^2 c_{\min}}{V^{1/3}} ;$$

*also, by using* $(*)$ *above, we have:*

$$\left( \frac{2}{\pi} \right)^{1/3} \frac{\Phi^2 \bar{C}}{V^{1/3}} \geq \left( \frac{2}{\pi} \right)^{1/3} \frac{\cancel{\Phi^2}}{V^{1/3}} \frac{|H|}{\cancel{\Phi^2}} = \left( \frac{2}{\pi} \right)^{1/3} \frac{|H|}{V^{1/3}} .$$

*Hence, from Theorem 2(i) follows (I). Moreover, at* $M_{\min}$*, from Theorem 1(ii) we have (II).*

■

- *In general, we have:*

$$\boxed{M_{\min} \propto c_{\min}}$$

*Question: do knot types of same* $c_{\min}$ *relax to same groundstate energy?*

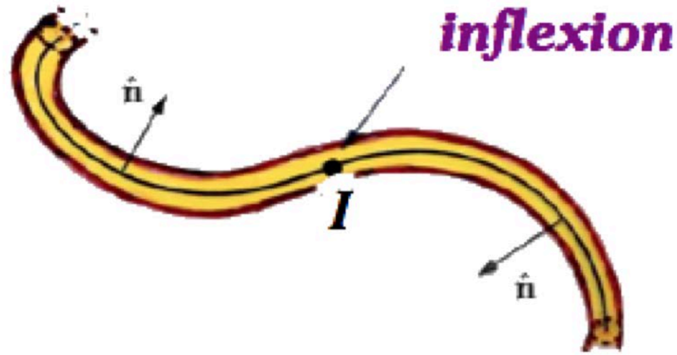
## *Inflexional magnetic knots*

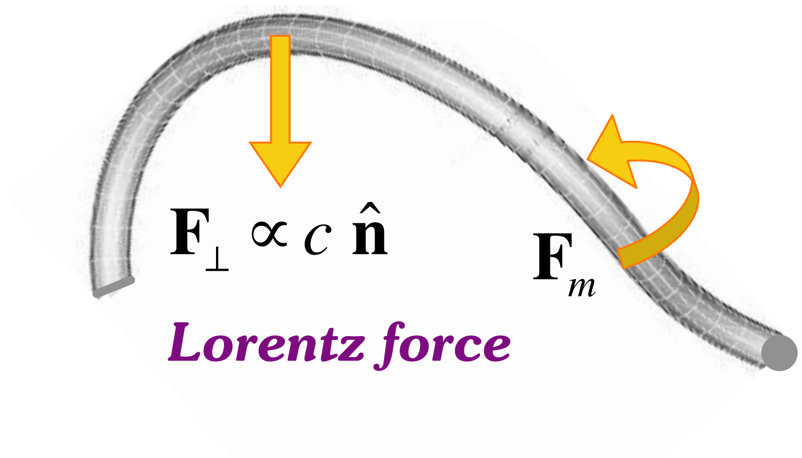
- ***Inflexion at I (in isolation):*** $c = 0$

  ***generic behaviour in*** $\mathbb{R}^3$***:***

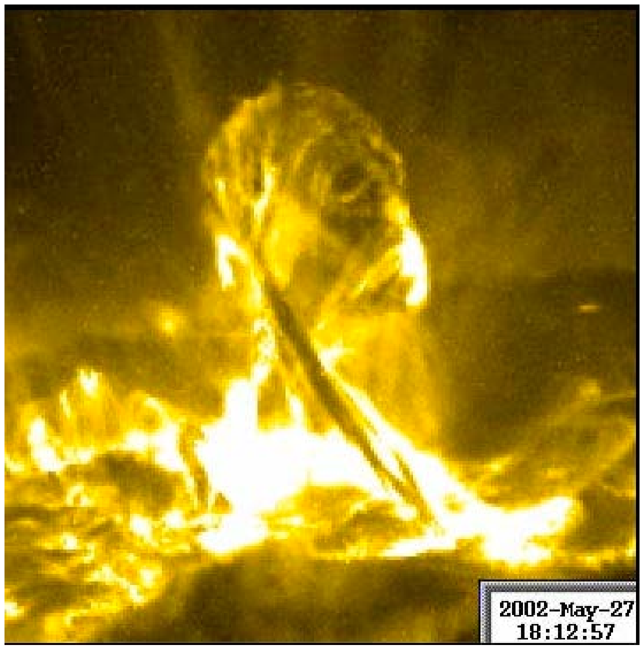
$$\mathbf{X}(s,t) = \left(s - \frac{2}{3}t^2 s^3, -ts^2, s^3\right)$$

***(Ricca & Moffatt 1992)***



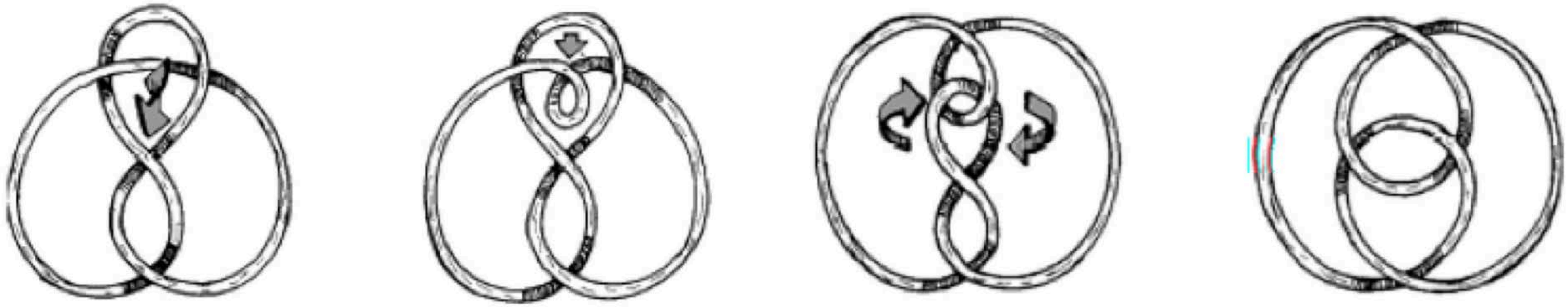

- ***Reidemeister type I move in action:***



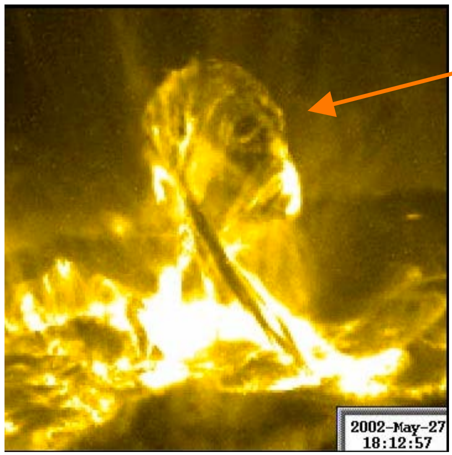

***(TRACE mission 2002)***

## From inflexion-free knots to magnetic braids

- **Definition.** *A knot in an inflexion-free configuration is a spiral knot.*

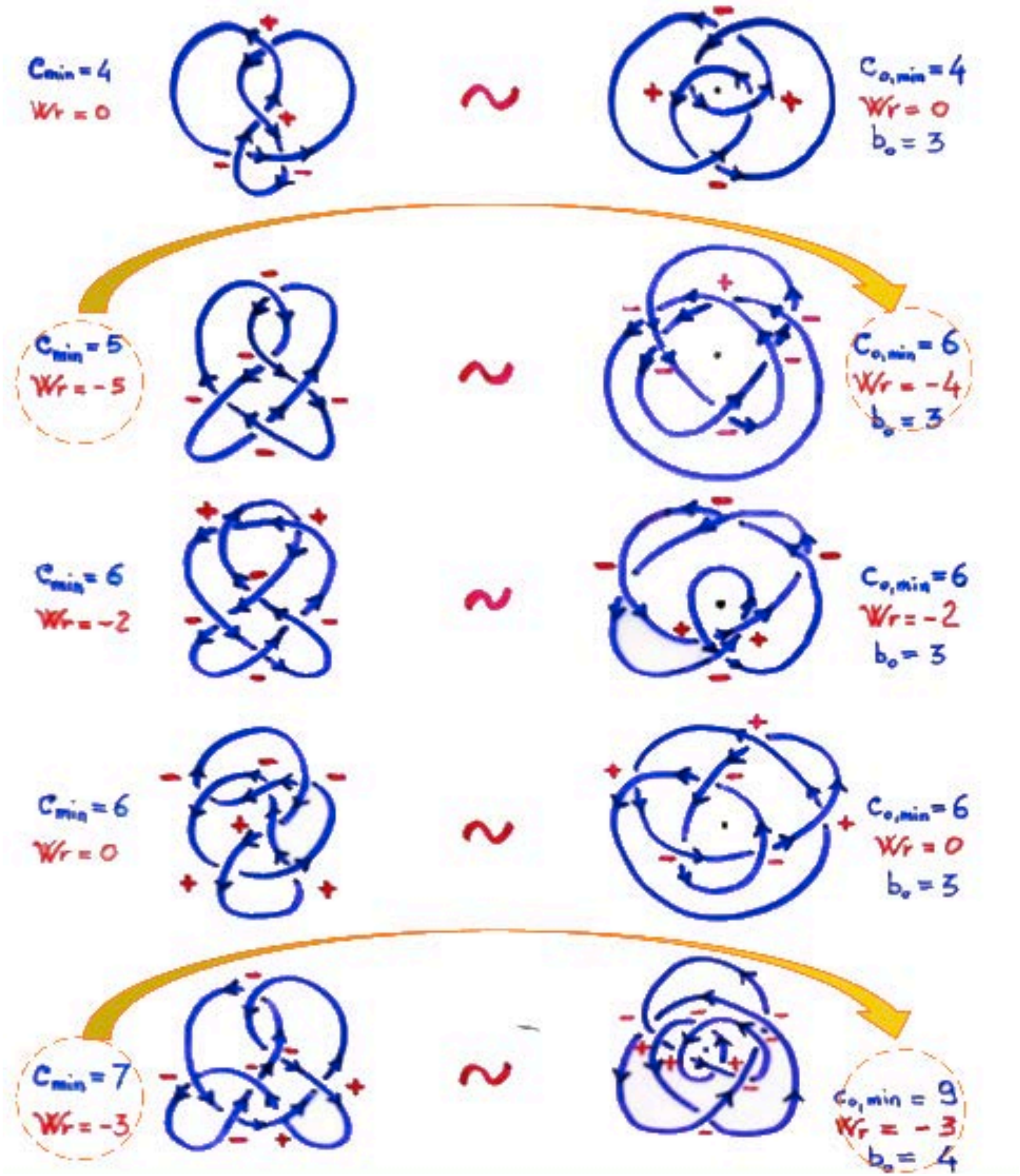


- **Theorem (Ricca 2005).** *Let $\tilde{\mathcal{K}}_{t_0}$ denote a loose magnetic knot in inflexional state. Then $\tilde{\mathcal{K}}_{t_0}$ is in inflexional disequilibrium.*
- **Corollary.** *If $\tilde{\mathcal{K}}_{t_0}$ is in inflexional disequilibrium, it relaxes to a spiral knot $\mathcal{K}_t$, for $t > t_0$ .*



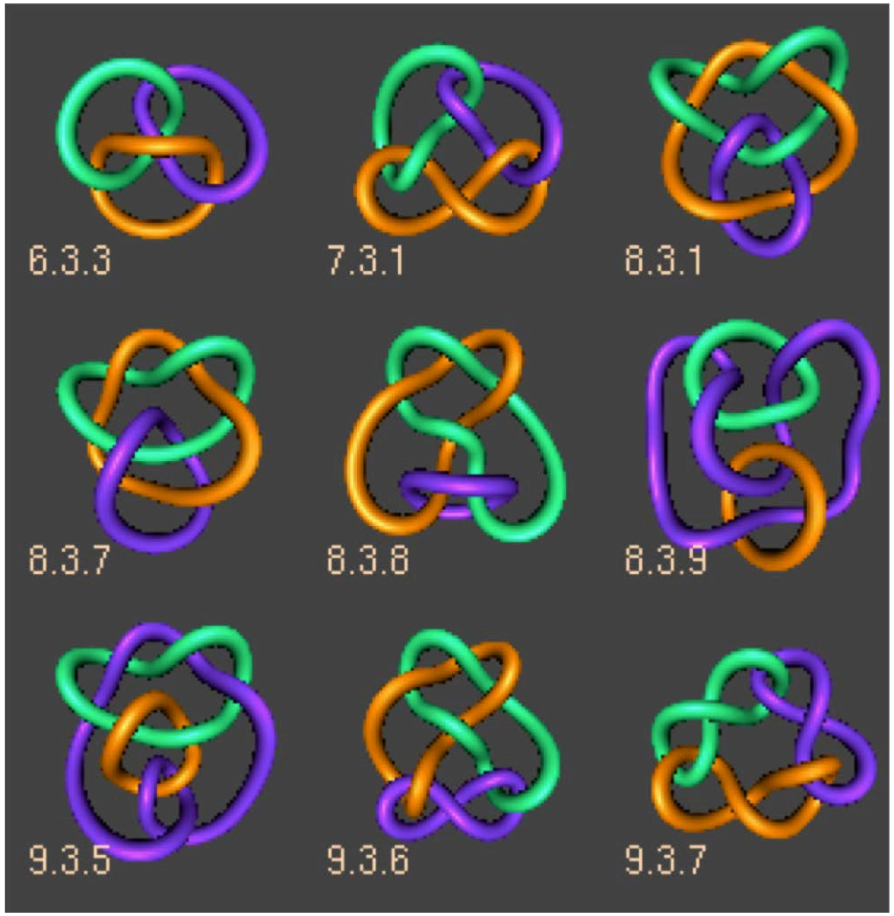

*inflexional instability*

# *From minimal knots to spiral knots*



# Knots and links tabulation

10-crossing knot table (Tait 1885)



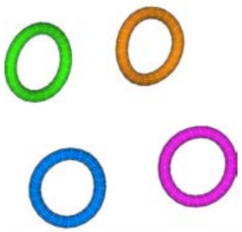

3-component links by `KnotPlot` (Scharein 2000)

...

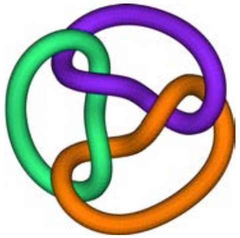
up to arbitrarily large # crossings $O(10^6)$

## *Groundstate energy of zero-framed knots and links*

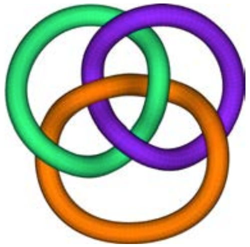
*Under signature-preserving flows, we have:* $\boxed{M_{\min} \propto c_{\min}}$



$0^4$

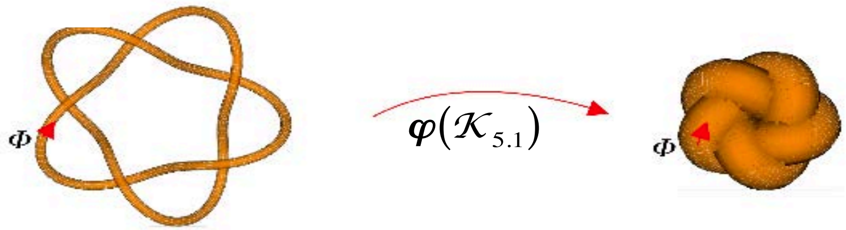


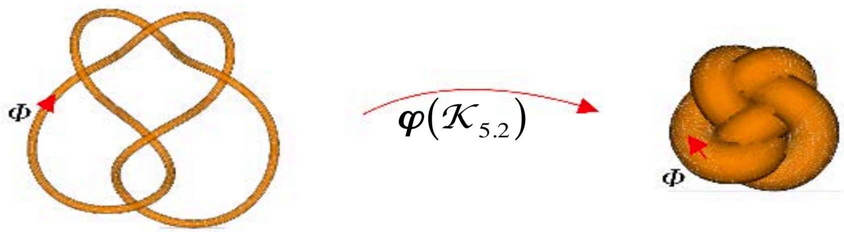
$6^3_1$



$6^3_2$

- ***Relaxation of 0-framed, 5-crossing knots:***



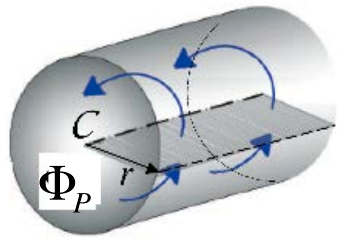




| V = 1, $\Phi$ = 1, 0-framing | |
|---|---|
| $c_{\min}$ | # knot types |
| 3 | 1 |
| 4 | 1 |
| 5 | 2 |
| 6 | 3 |
| 7 | 7 |
| 8 | 21 |
| 9 | 49 |
| 10 | 166 |
| … | … |

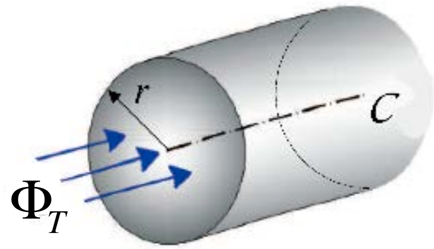
## Constrained minimization of magnetic energy of knots

- ***tubular knot*** $K$ **:** $V(K) = \pi a^2 L$ **; *Mercier (orthogonal) system:*** $(r,\vartheta,s)$

- ***magnetic field:*** $\mathbf{B} = (0,\ B_\vartheta(r),\ B_s(r)) \qquad (\nabla\cdot\mathbf{B}=0)$




$$\mathbf{B} = \left(0, \frac{1}{L}\frac{\mathrm{d}\Phi_P}{\mathrm{d}r}, \frac{1}{2\pi r}\frac{\mathrm{d}\Phi_T}{\mathrm{d}r}\right) + \left(0, \frac{\partial\tilde{\psi}}{\partial s}, -\frac{\partial\tilde{\psi}}{\partial\vartheta}\right)$$




- ***fluxes*** $\Phi_P$ **,** $\Phi_T$**:**

***twist parameter:*** $h = \Phi_P / \Phi_T$

- ***Theorem (Maggioni & Ricca 2009). Let us assume that***

  ***(i)*** $\{V, \Phi\}$ ***invariant*** $(V = 1, \Phi = 1)$***;***

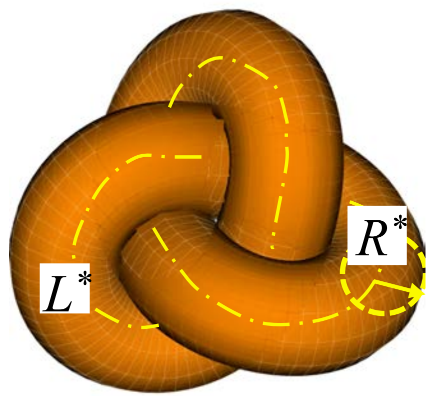
  ***(ii) circular cross-section independent of arc-length;***

  ***(iii)*** $\tilde{\psi}$ ***independent of arc-length;***

  ***(iv)*** $L$ ***independent of internal twist.***

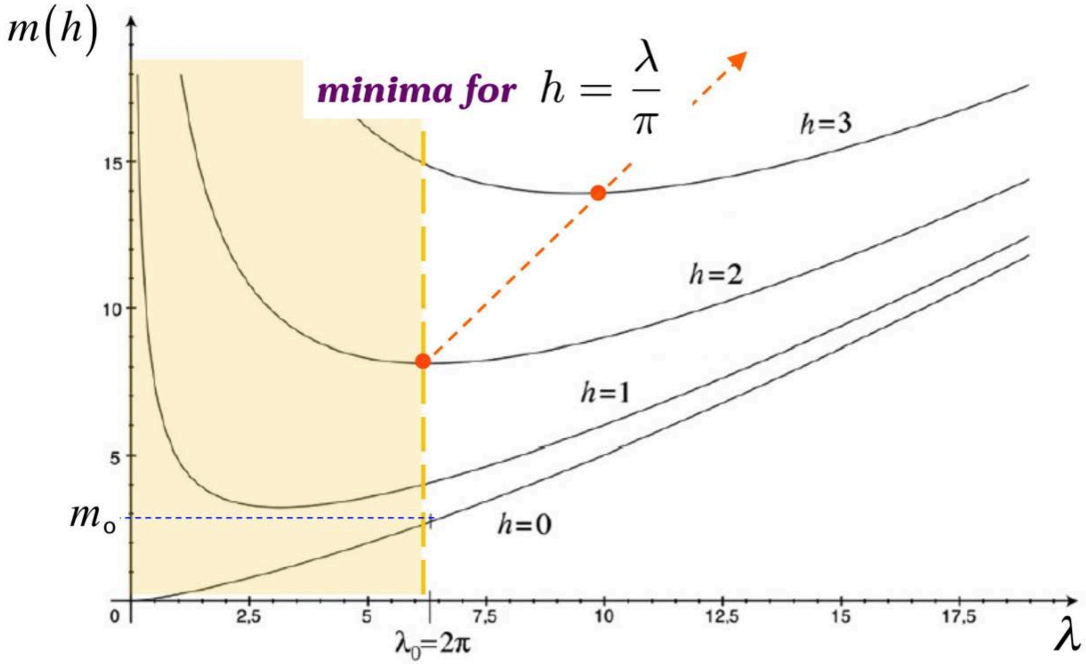
  ***Then, we have***

$$M_\lambda^*(h) = \frac{\lambda^{4/3}}{2\pi^{2/3}} + \frac{\pi^{4/3}h^2}{\lambda^{2/3}} = m(\lambda, h) \quad .$$



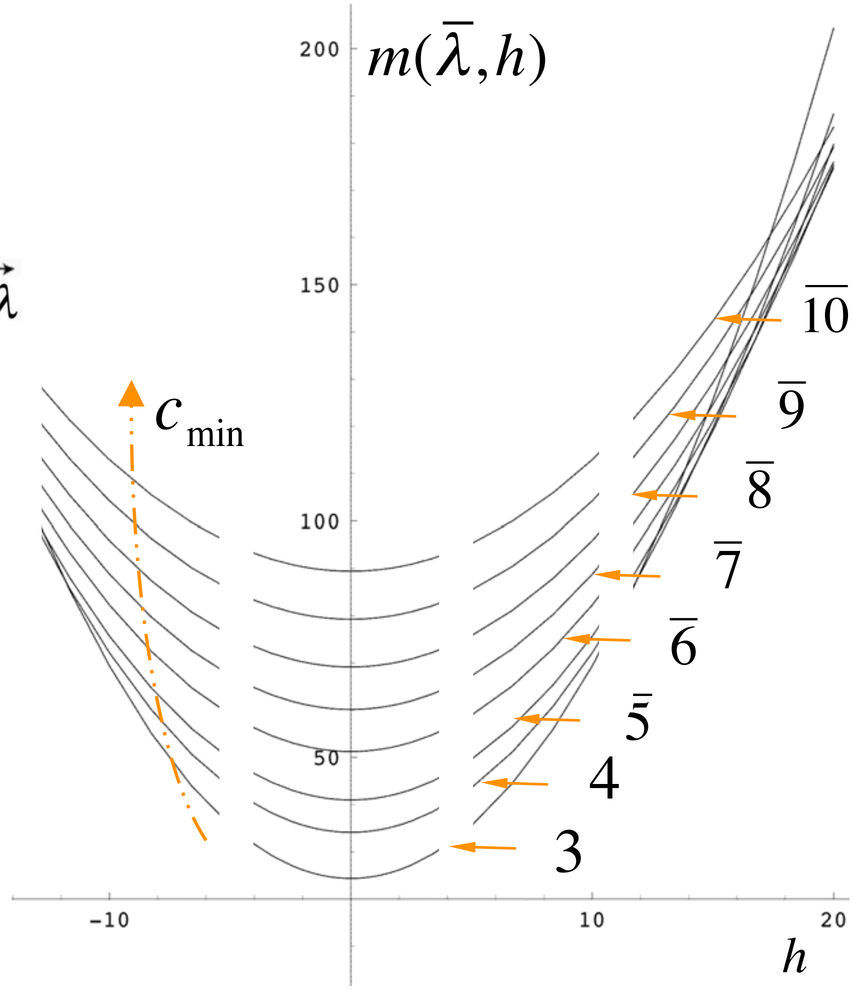

***ropelength:*** $\lambda = L^*/R^*$

## *Groundstate energy spectrum: averaging over complexity*




$$m_{\min}(h) = \frac{3}{2}\pi^{2/3}h^{4/3} \qquad (h \geq 2)$$

***(see also Chui & Moffatt 1992)***




***tightening:*** $V = 1$, $\Phi = 1$, $h = 0$

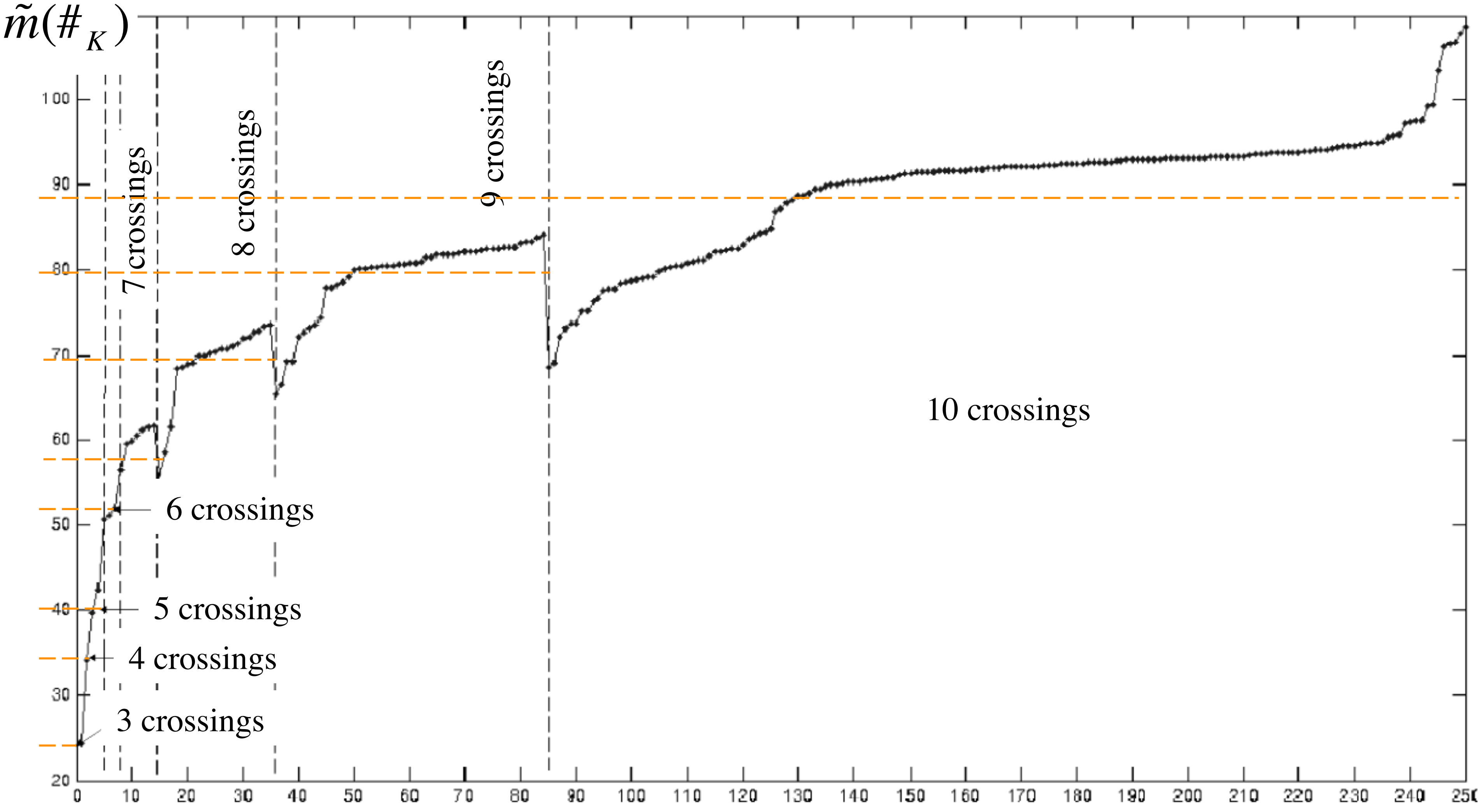
**SONO** ***(Pieranski et al. 1998)***

**RIDGERUNNER** ***(Ashton et al. 2010)***

# Groundstate energy spectrum of first 250 prime knots

- $V = 1$, $\Phi = 1$, $h = 0$
- ***tight unknot:*** $M_o^* = \left(2\pi^2\right)^{1/3}$

$$\tilde{m}(\#_K) = \frac{m\left(\lambda(\#_K),0\right)}{M_o^*} = \left(\frac{\lambda(\#_K)}{2\pi}\right)^{4/3}$$

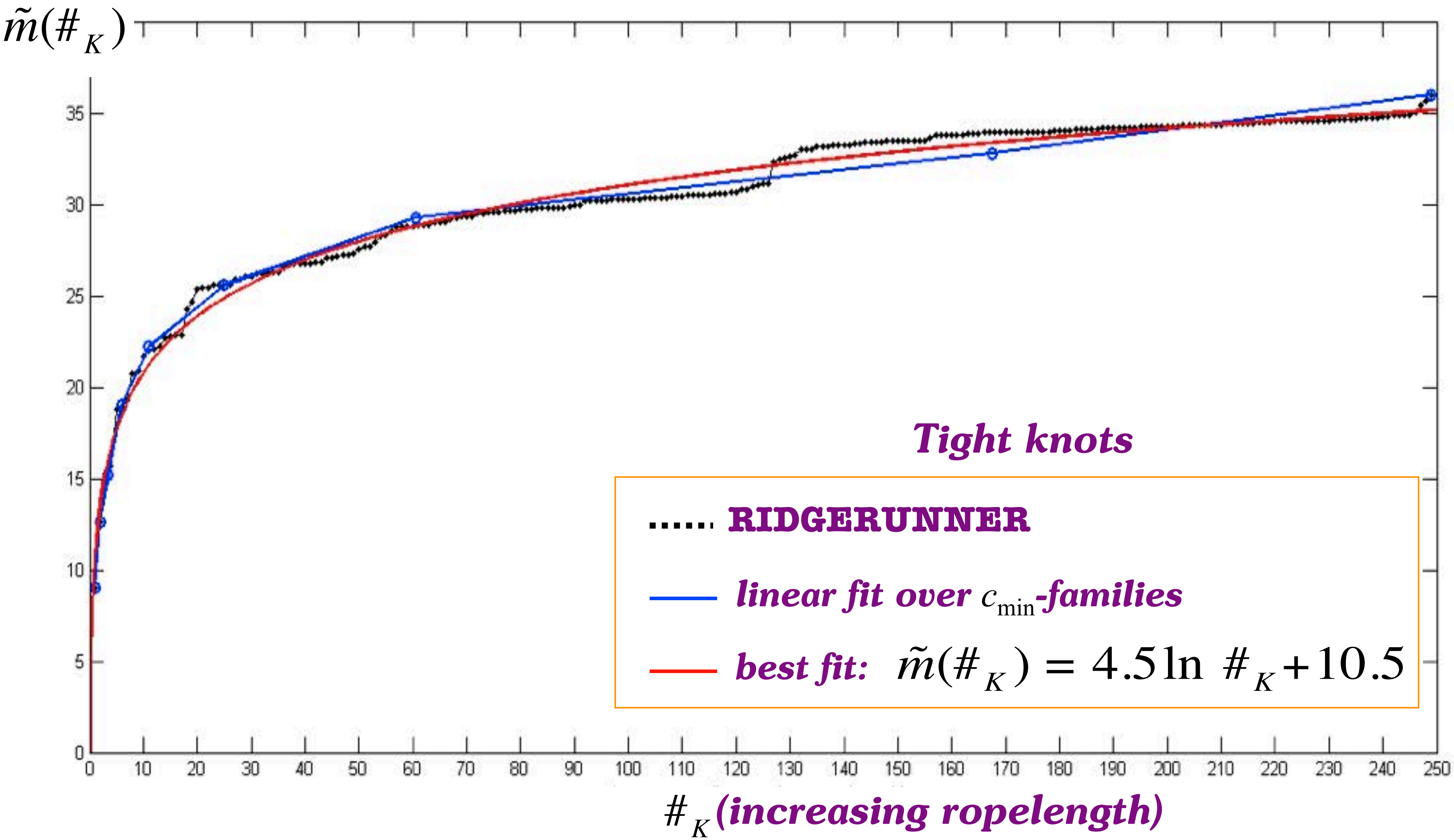



$\#_K$ ***(Rolfsen table)***

## Knot energy spectrum by increasing ropelength (Ricca & Maggioni 2014)

- $V = 1$, $\Phi = 1$, $h = 0$
- **tight unknot:** $M_o^* = \left(2\pi^2\right)^{1/3}$

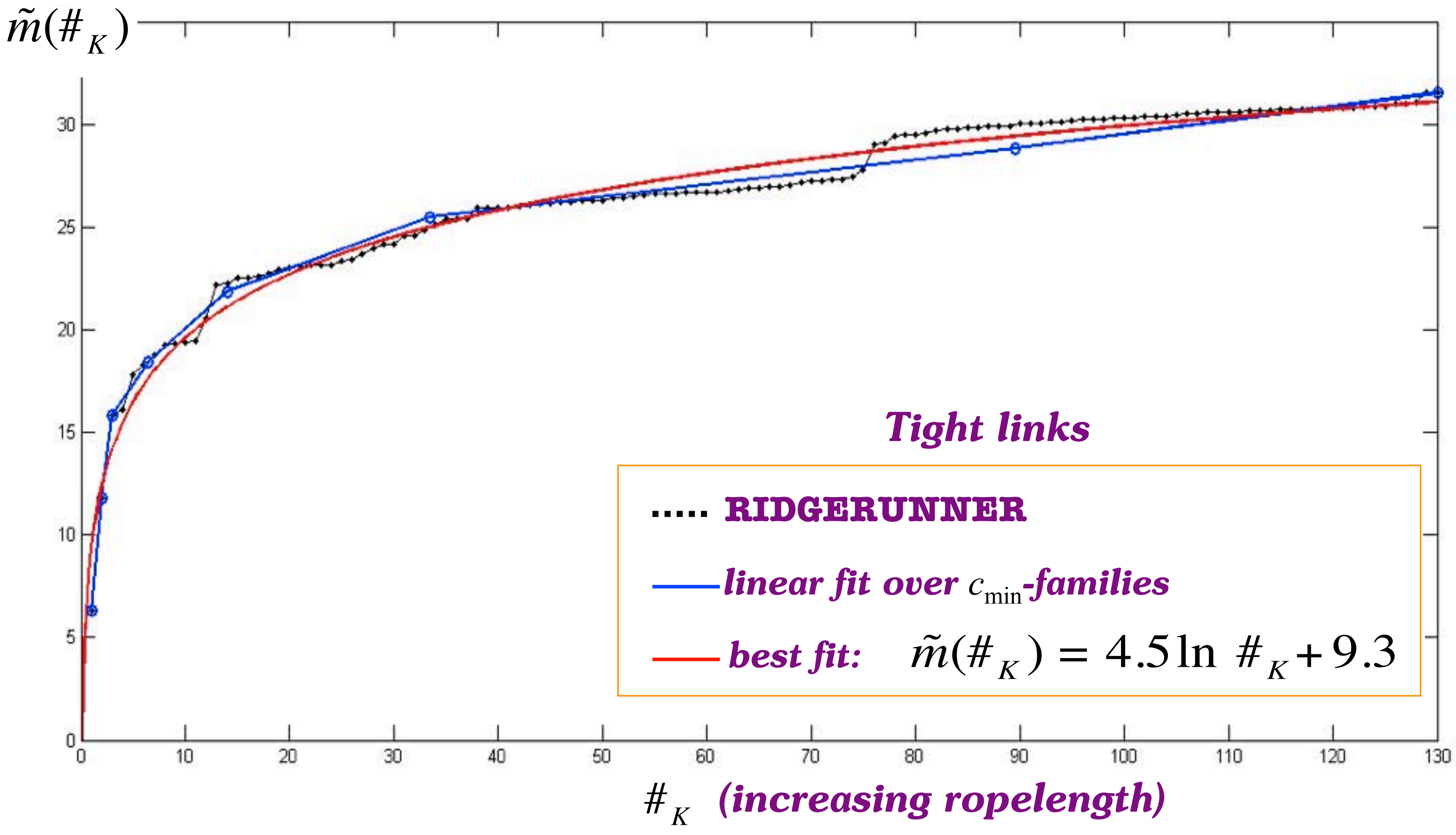
$$\tilde{m}(\#_K) = \frac{m\left(\lambda(\#_K),0\right)}{M_o^*} = \left(\frac{\lambda(\#_K)}{2\pi}\right)^{4/3}$$



## *Link energy spectrum by increasing ropelength* (Ricca & Maggioni 2014)

- $V = 1$, $\Phi = 1$, $h = 0$
- *tight unknot:* $M_o^* = \left(2\pi^2\right)^{1/3}$

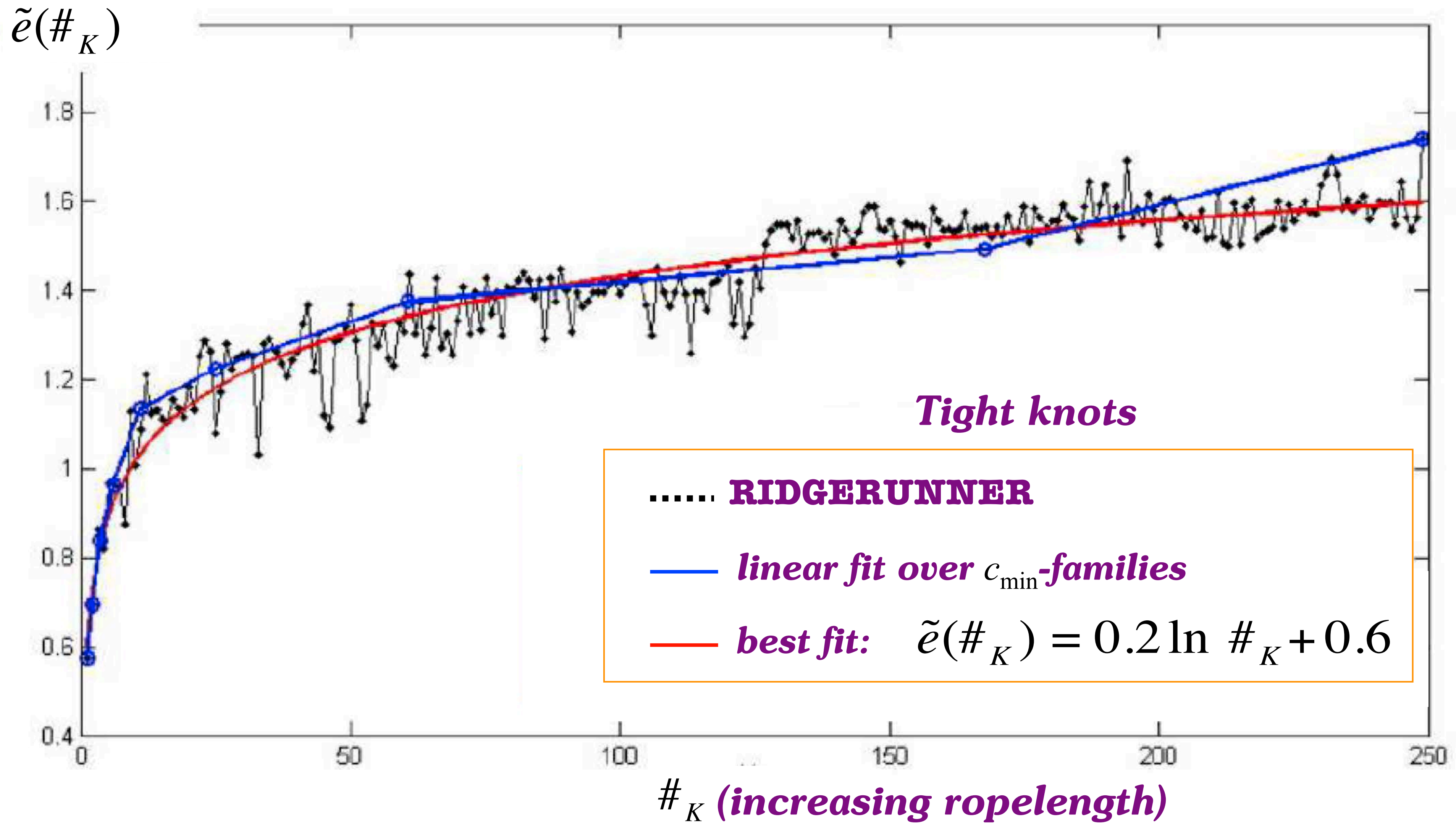
$$\tilde{m}(\#_K) = \frac{m\left(\lambda(\#_K),0\right)}{M_o^*} = \left(\frac{\lambda(\#_K)}{2\pi}\right)^{4/3}$$



## *Bending energy spectrum of elastic knots*

- $V = 1$, $h = 0$
- ***tight torus:*** $E_o = \pi R^* 2^{1/3} \pi^{5/3}$

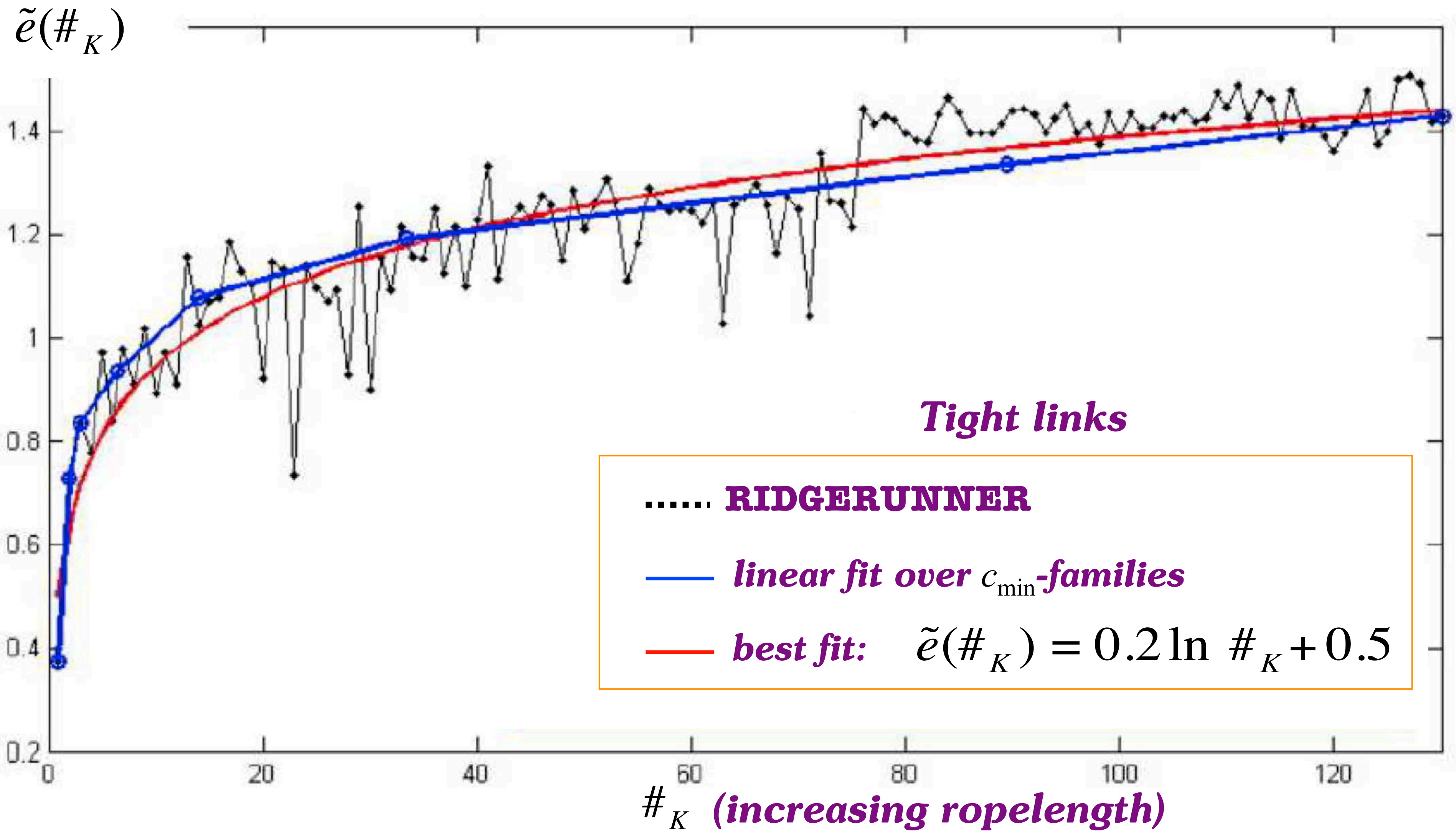
$$\tilde{e} = \frac{E_b}{E_o} = \frac{\int_K [c(s)]^2 ds}{2^{4/3}\pi^{5/3}}$$



# Bending energy spectrum of elastic links

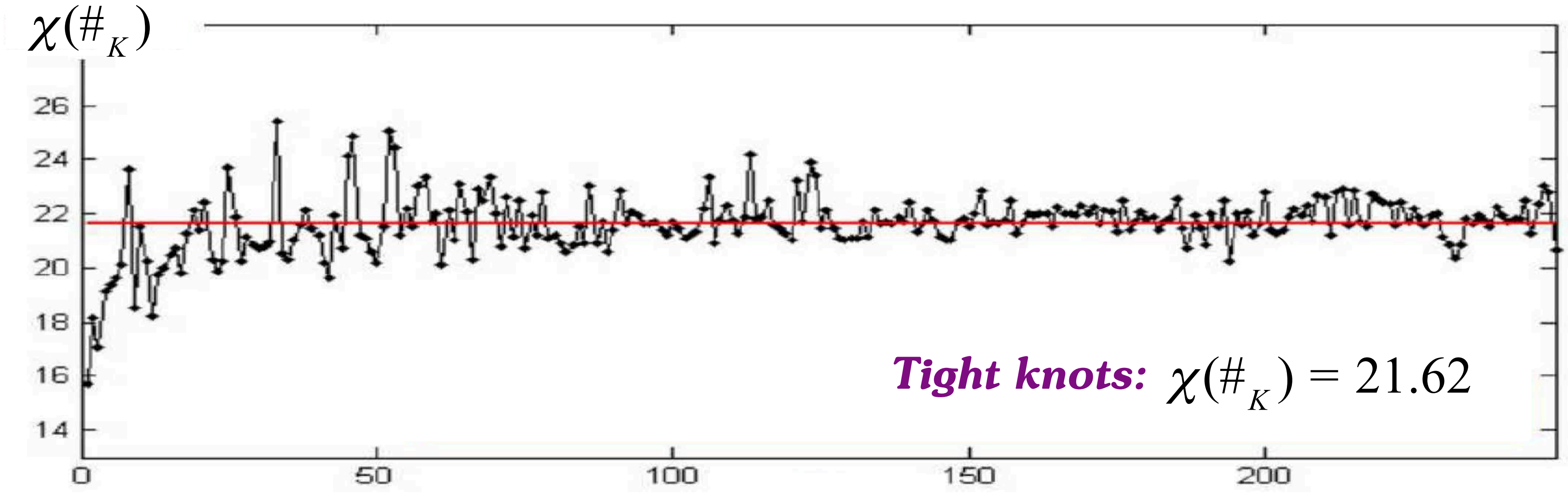
- $V = 1$, $h = 0$
- *tight torus:* $E_o = \pi R^* 2^{1/3} \pi^{5/3}$

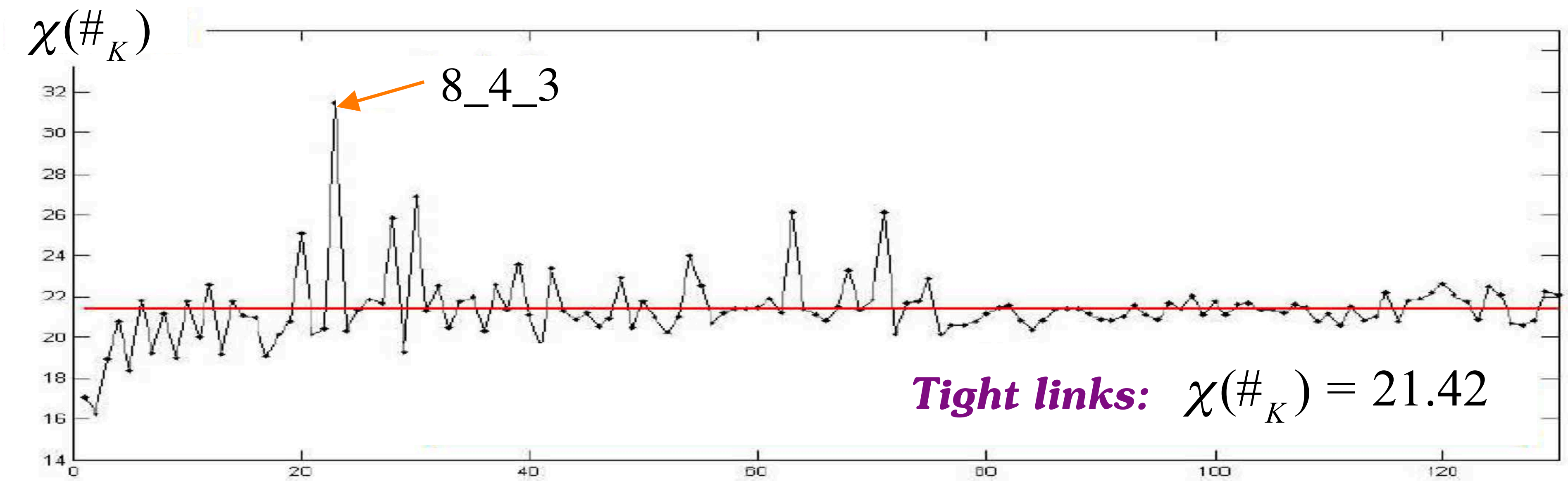
$$\tilde{e} = \frac{E_b}{E_o} = \frac{\int_K [c(s)]^2 ds}{2^{4/3} \pi^{5/3}}$$



## *Magnetic versus bending energy*

***(Ricca & Maggioni 2017)***




$\#_K$ ***(increasing ropelength)***

## *Ropelength versus topological crossing number*

- ***Assumption:*** $\#_K \sim A^{c_{\min}}$ ***, then from*** $\tilde{m}(\#_K) \propto \left[\lambda(\#_K)\right]^{4/3}$ ***we have*** $\left[\lambda(\#_K)\right]^{4/3} \propto a \ln \#_K + b$ ***; then*** $\left[\lambda(\#_K)\right]^{4/3} \propto c_{\min}$ ***, or***

$$\lambda(\#_K) \propto c_{\min}^{3/4} \ ,$$

***in partial agreement with***

$$O\left(c_{\min}^{3/4}\right) \le \lambda(\#_K) \le O\left(c_{\min} \ln^5 c_{\min}\right) .$$

***(Buck & Simon 1999; Cantarella et al. 2002; Diao 2003; Diao et al. 2013).***

- ***From lower bounds on energy*** $(V = 1,\ \Phi = 1)$***, we have:***

$$M_{\min} = \left(\frac{2}{\pi}\right)^{1/3} c_{\min} \qquad \textbf{and} \qquad \bar{m}\left(c_{\min}\right) \equiv \frac{M_{\min}}{m_o} = \frac{c_{\min}}{\pi} \ ;$$

***since***

$$\left\langle \tilde{m}\left(\#_K\right)\right\rangle \ge \bar{m}\left(c_{\min}\right) = \frac{c_{\min}}{\pi} \ ,$$

***then***

$$\left\langle \lambda\left(\#_K\right)\right\rangle \ge 2\pi^{1/4} c_{\min}^{3/4} \approx \underline{2.66}\ c_{\min}^{3/4} \qquad \forall\ c_{\min} \ ,$$

***better than Buck & Simon (1999), where constant*** $(4\pi/11)^{3/4} \approx \underline{1.10}$ .