Lecture 2

- ***Vortex knots dynamics and momenta of a tangle:***

    - Localized Induction Approximation (LIA) and Non-Linear Schrödinger (NLS) equation

    - Integrable vortex dynamics and LIA hierarchy

    - Torus knot solutions to LIA

    - Linear and angular momentum in terms of signed area information

- ***Selected references***

***Maggioni, F, Alamri, SZ, Barenghi, CF & Ricca, RL 2010** Velocity, energy and helicity of vortex knots and unknots. Phys Rev E* 82, 26309.

***Ricca, RL 1993** Torus knots and polynomial invariants for a class of soliton equations. Chaos* 3, 83 [Erratum. *Chaos* 5, 346].

***Ricca, RL 2008** Momenta of a vortex tangle by structural complexity analysis. Physica D* 237, 2223.

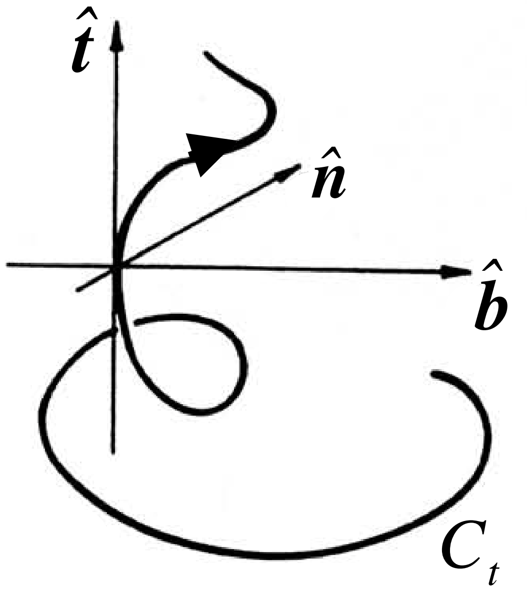
Localized induction approximation (LIA)

homogeneous
incompressible
inviscid
} fluid in $\mathbb{R}^3$:

$$\boldsymbol{u} = \boldsymbol{u}(X,t) \quad \begin{cases} \boldsymbol{\omega} = \nabla \times \boldsymbol{u} \\ \nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \mathbb{R}^3 \\ \boldsymbol{u} = 0 \quad \text{as } X \to \infty \end{cases}$$

- **Space curve C_t, given by:**

$$C_t : \begin{cases} X(s,t) := X_t(s) \in C^{\nu} \\ s \in [0,L] \to \mathbb{R}^3 \end{cases}$$



- **Intrinsic reference on C_t, given by:**
(Frenet frame)

$$\hat{\boldsymbol{t}} := X'(s,t) = \frac{\partial X}{\partial s} \qquad \{\hat{\boldsymbol{t}}, \hat{\boldsymbol{n}}, \hat{\boldsymbol{b}}\}$$

- **Vortex line on C_t :**

$\boldsymbol{\omega} = \omega_0 \hat{\boldsymbol{t}}$, $\omega_0 = \text{constant}$

circulation $K = \text{constant}$

asymptotic theory: $R/a >> 1$

$$\dot{X}(s,t) = \frac{\partial X}{\partial t} = \underline{\underline{X' \times X'' = c\hat{\boldsymbol{b}} = \boldsymbol{u}_{\text{LIA}}}}$$

(Da Rios 1906; Hama & Arms 1961)

Intrinsic equations under LIA and NonLinear Schrödinger (NLS) equation

- ***Intrinsic description:*** $\boldsymbol{u} = (u_t, u_n, u_b)$, $c(s,t)$ ***curvature,*** $\tau(s,t)$ ***torsion;***

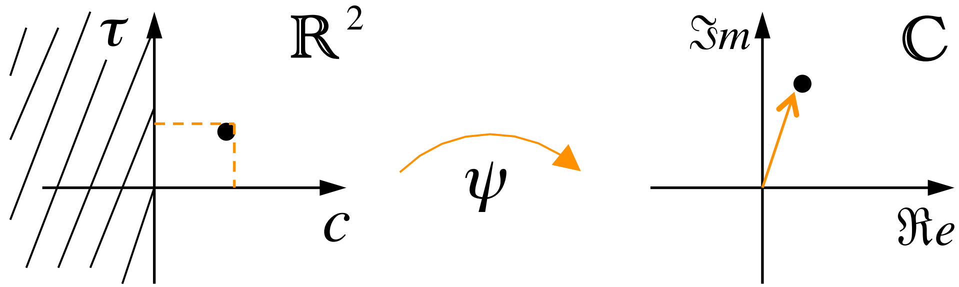
  under LIA: $\boldsymbol{u}_{\mathrm{LIA}} = \mathrm{c}\hat{\boldsymbol{b}}$, $u_t = u_n = 0$, $u_b = \mathrm{c}$.

$$C_t := \varphi_t(C) \quad \begin{cases} \dot{c} = -(c\tau)' - c'\tau \\ \dot{\tau} = \left(\dfrac{c'' - c\tau^2}{c} \right)' + c'c \end{cases}$$

(Da Rios 1906 Betchov 1965)

- ***NLS eq. via Madelung transform:***

$$\psi(s,t) = c(s,t)\ \mathrm{e}^{\mathrm{i}\int_0^s \tau(\xi,t)\mathrm{d}\xi}$$



(Hasimoto 1972)

NLS eq.: $i\dot{\psi} + \psi'' + \frac{1}{2}|\psi|^2\psi = 0$

- ***Intrinsic equations by log-derivative:***

  let $\Theta(s,t) = \int_0^s \tau(\xi,t)\mathrm{d}\xi$ ***and consider*** $\psi = c\ \mathrm{e}^{\mathrm{i}\Theta}$ ⇒ ***Da Rios-Betchov eqs.***

Integrable vortex dynamics and LIA hierarchy

- **Integrable vortex dynamics:**

$$\boldsymbol{u} = (\alpha + \beta s)c\hat{\boldsymbol{b}} + (\gamma + \delta s)\hat{\boldsymbol{t}} + \mu\left(\tfrac{1}{2}c^2\hat{\boldsymbol{t}} + c'\hat{\boldsymbol{n}} + c\tau\hat{\boldsymbol{b}}\right)$$

β, δ **: inhomogeneities**
(Lakshmanan & Ganesan 1985)

μ **: axial flow and vorticity**
(Fukumoto & Miyazaki 1991)

γ **: non-linear stretching**
(Onuki 1985)

- **LIA hierarchy:**

$$\begin{cases} \boldsymbol{u}^{(0)} = \boldsymbol{u}_{\mathrm{LIA}} \\ \boldsymbol{u}^{(j+1)} = \dot{\hat{\boldsymbol{t}}} \times \hat{\boldsymbol{t}} + \left[\int c(s,t)\left(u_b'^{(j)} + \tau u_n^{(j)}\right)\mathrm{d}s\right]\hat{\boldsymbol{t}} = \sum_{\nu=1}^{j+1} \mathscr{F}\left(\dfrac{\partial^\nu \boldsymbol{u}^{(0)}}{\partial s^\nu}\right) = \mathscr{R}\left(\boldsymbol{u}^{(0)}\right) \end{cases}$$

(Langer & Perline 1991)

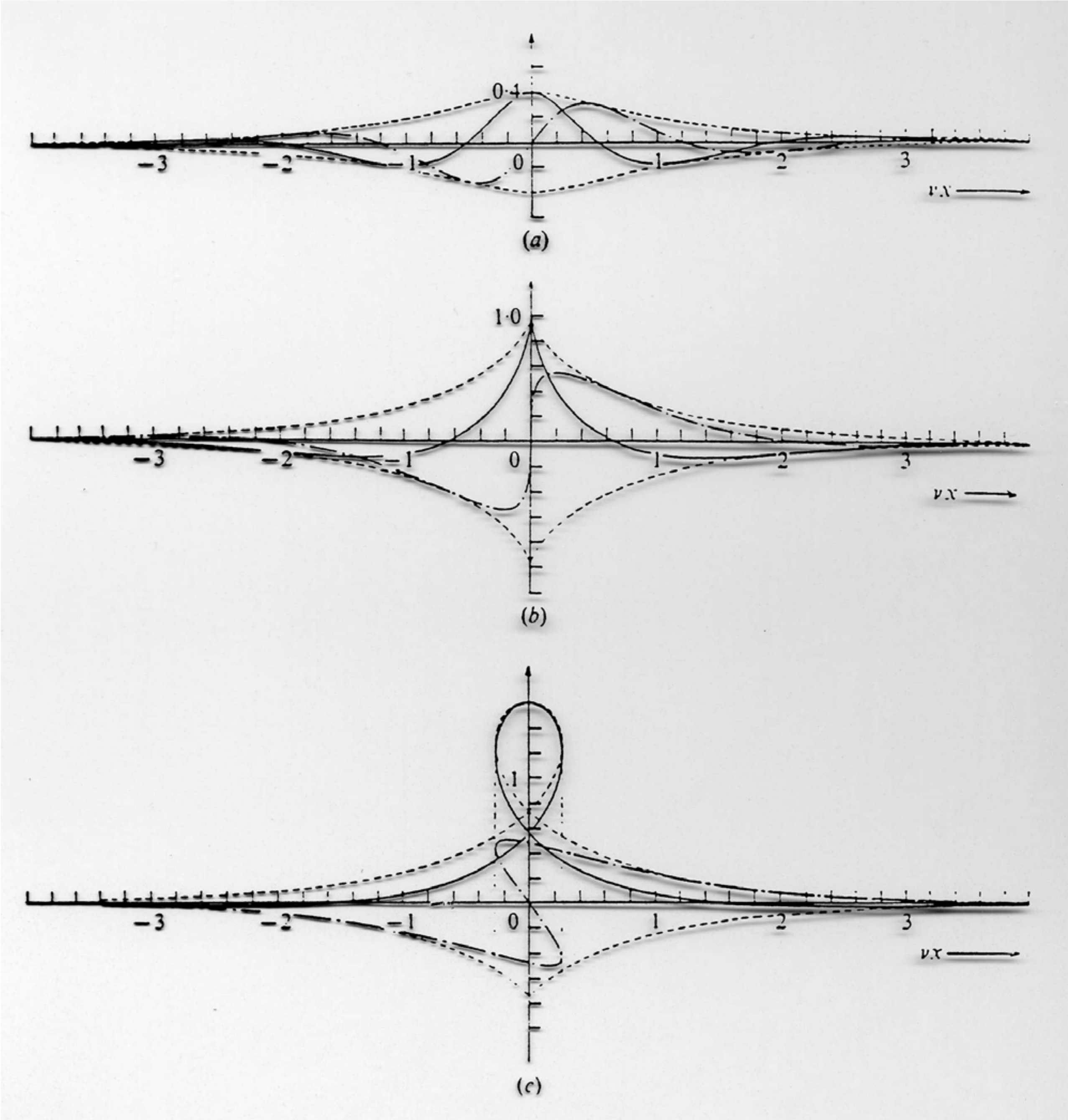
- **integrable class:**

$$\dot{\psi} = \mathscr{P}_\psi\left(u_n \mathrm{e}^{\mathrm{i}\Theta}\right) + \mathscr{Q}_\psi\left(u_b \mathrm{e}^{\mathrm{i}\Theta}\right)$$

(Nakayama *et al.* 1992)

$$\begin{cases} \textbf{Hirota class} \begin{cases} \textbf{NLSE} \\ \textbf{mKdV} \\ \textbf{sine-G} \\ \dots\ \dots \end{cases} \\ \dots\ \dots\ \dots \end{cases}$$

Stationary solutions: Hasimoto soliton

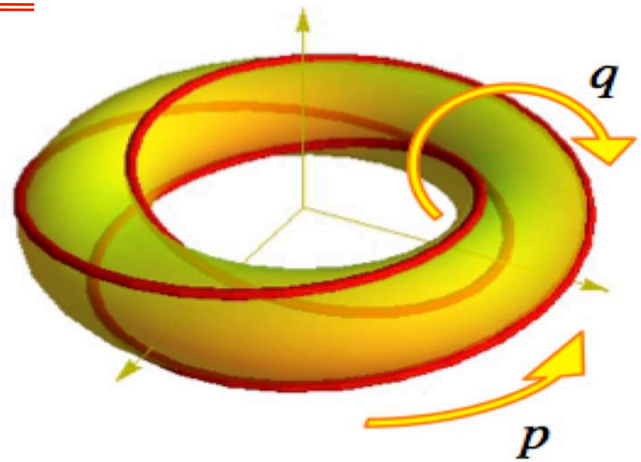


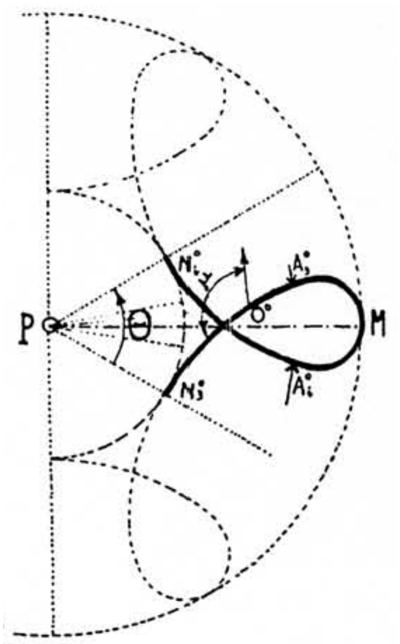
$$\begin{cases} \tau = \text{const.} \\ c(s) = 2\,\text{sech}(s) \end{cases}$$

(Hasimoto 1972)

Steady solutions to LIA: torus knots and unknots

Look for stationary solutions to LIA in the shape of torus knots and unknots:





(Da Rios 1933)

- **Theorem** *(Kida 1981). There exists a class of steady torus knot solutions $\mathcal{T}_{p,q}$ ($p>1$, $q>1$ co-prime integers) to LIA in terms of incomplete elliptic integrals, given by*

$$r^2 = F_r(J)\,, \quad \alpha = F_\alpha(E)\,, \quad z = F_z(\Pi)\ .$$

- **Theorem** *(Ricca 1993). LIA admits torus knot solutions $\mathcal{T}_{p,q}$ in closed analytic form, given by:*

$$\begin{cases} r = r_0 + \varepsilon \sin(w\xi) \\ \alpha = \dfrac{s}{r_0} + \dfrac{\varepsilon}{w r_0}\cos(w\xi) \\ z = \dfrac{\bar{t}}{r_0} + \varepsilon\left(1 - \dfrac{1}{w^2}\right)^{1/2}\cos(w\xi)\ . \end{cases}$$

Proof. ***First let's write LIA in terms of cartesian coordinates*** (x, y, z) ***and cylindrical polar coordinates*** (r, α, z).

From

$$\boldsymbol{u}_{\mathrm{LIA}} = \dot{\boldsymbol{X}}(s,t) = \boldsymbol{X}' \times \boldsymbol{X}'' \qquad (*)$$

and $x = r\cos\alpha$, $y = r\sin\alpha$, z $(**)$, ***by taking time-derivative, we have***

$$\begin{cases} \dot{x} = \dot{r}\cos\alpha - r\sin\alpha\dot{\alpha} \\ \dot{y} = \dot{r}\sin\alpha + r\cos\alpha\dot{\alpha} \\ \dot{z} = \dot{z} \end{cases} \Longrightarrow \begin{cases} \dot{r} = \dot{x}\cos\alpha + \dot{y}\sin\alpha \\ r\dot{\alpha} = -\dot{x}\sin\alpha + \dot{y}\cos\alpha \\ \dot{z} = \dot{z} \end{cases}$$

and by the last equality of $(*)$, ***we have***

$$\begin{cases} \dot{r} = (y'z'' - y''z')\cos\alpha + (z'x'' - z''x')\sin\alpha \\ r\dot{\alpha} = (z'y'' - z''y')\sin\alpha + (z'x'' - z''x')\cos\alpha \\ \dot{z} = x'y'' - x''y' \ . \end{cases}$$

We now take the arc-length derivative of $(**)$ ***and substitute in the above, to get LIA in cylindrical polar coordinates:***

$$\begin{cases} \dot{r} = r\alpha'z'' - 2r'\alpha'z' - r\alpha''z' \\ r\dot{\alpha} = -r'z'' + r''z' - r\alpha'^2 z' \\ \dot{z} = 2r'^2\alpha' - rr'\alpha'' - rr''\alpha' + r^2\alpha'^3 \ . \end{cases} \qquad (***)$$

Now let's consider small-amplitude perturbations. For the vortex ring we have $r = R$, ***and*** $\alpha = s/R$, ***so that previous equations reduce to***

$$\dot{z} = r^2 \alpha'^3 = R^2 \frac{1}{R^3} = \frac{1}{R} \quad \Longrightarrow \quad z = \frac{t}{R} \ .$$

Small-amplitude perturbations are given by taking

$$r = R + \epsilon r_1, \quad \alpha = \frac{s}{R} + \epsilon \alpha_1, \quad z = \frac{t}{R} + \epsilon z_1 \ .$$

By substituting these into $(***)$ ***and taking first-order terms, we have***

$$\begin{cases} \dot{r}_1 = z_1'' \\ \dot{\alpha}_1 = -\frac{1}{R^2} z_1' \\ \dot{z}_1 = -r_1'' - \frac{1}{R^2} r_1 \end{cases}$$

By looking for a traveling-wave solution $\xi = s - \kappa t$ ***as torus knot, we have***

$$\begin{cases} r = R + \epsilon \kappa_r \sin\left(w\frac{\xi}{R} + \beta_k\right) \\ \alpha = \frac{s}{R} + \epsilon \frac{\kappa_r}{wR} \cos\left(w\frac{\xi}{R} + \beta_k\right) \\ z = \frac{t}{R} + \epsilon \kappa_r \sqrt{\frac{w^2-1}{w^2}} \cos\left(w\frac{\xi}{R} + \beta_k\right) \end{cases} \quad \Longrightarrow \quad \begin{cases} r = r_0 + \varepsilon \sin(w\xi) \\ \alpha = \dfrac{s}{r_0} + \dfrac{\varepsilon}{wr_0} \cos(w\xi) \\ z = \dfrac{\bar{t}}{r_0} + \varepsilon \left(1 - \dfrac{1}{w^2}\right)^{1/2} \cos(w\xi) \ . \end{cases}$$

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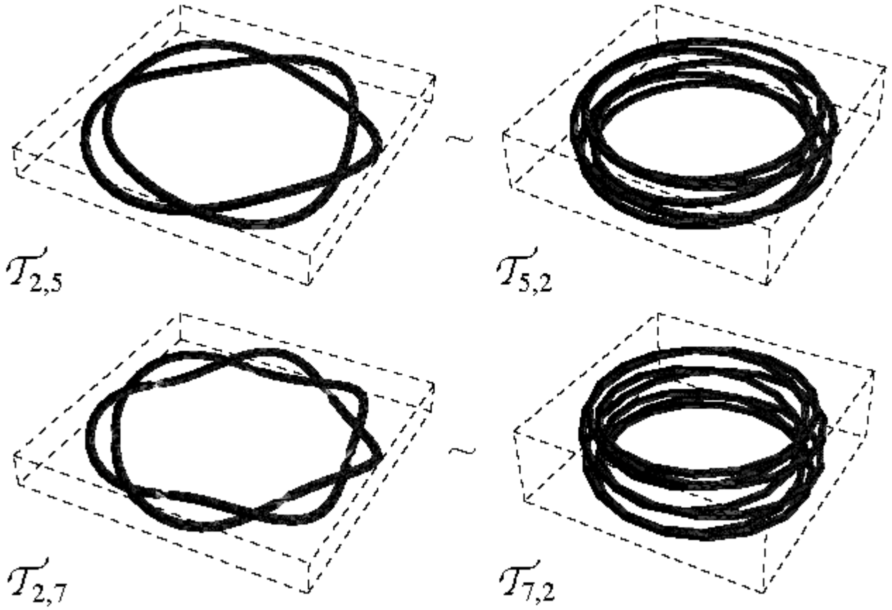
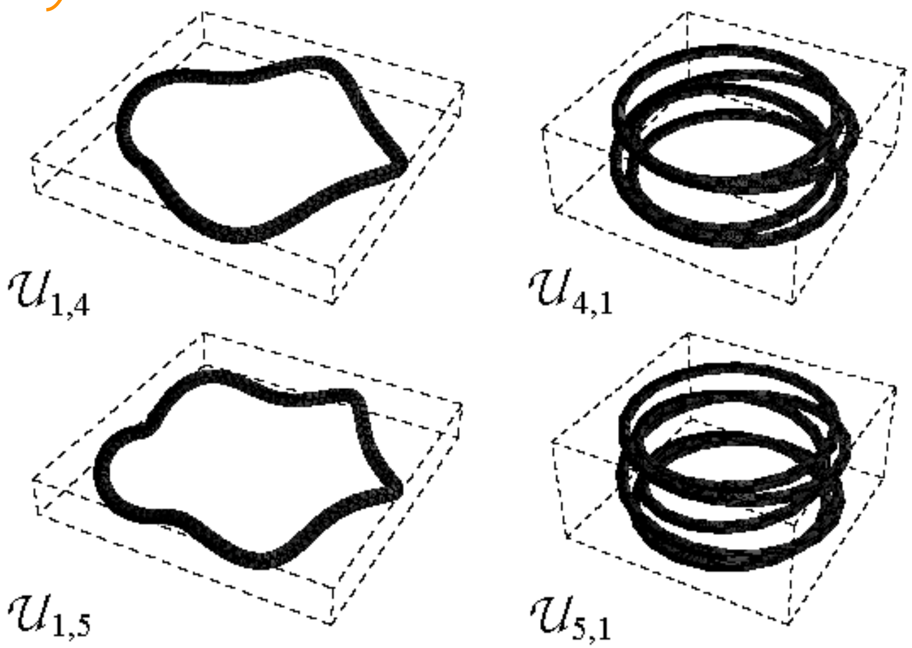
Small-amplitude torus knots and unknots under LIA



torus knots

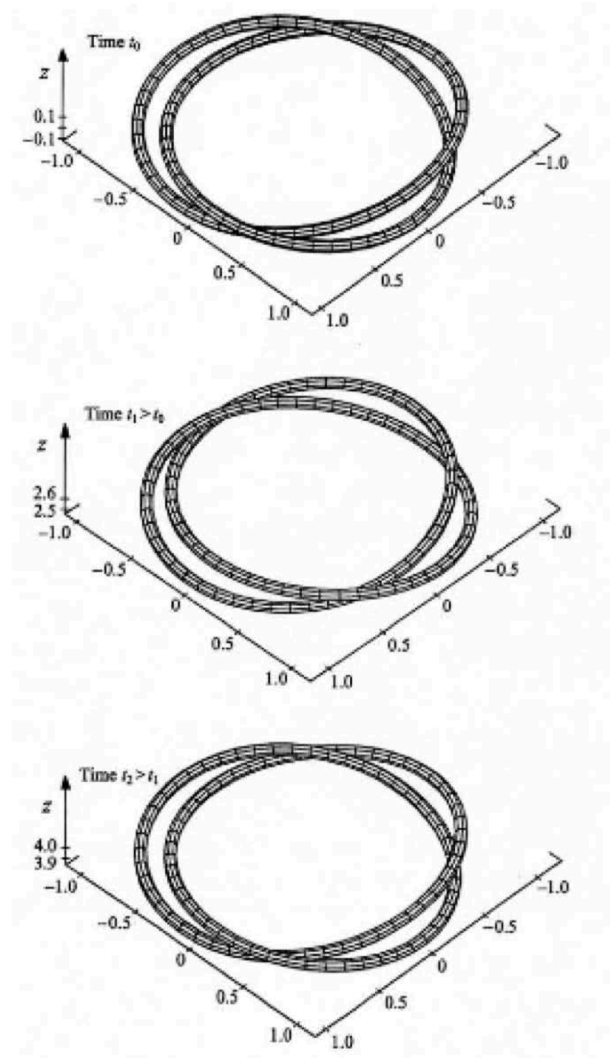
torus unknots



Linear stability of LIA torus knots

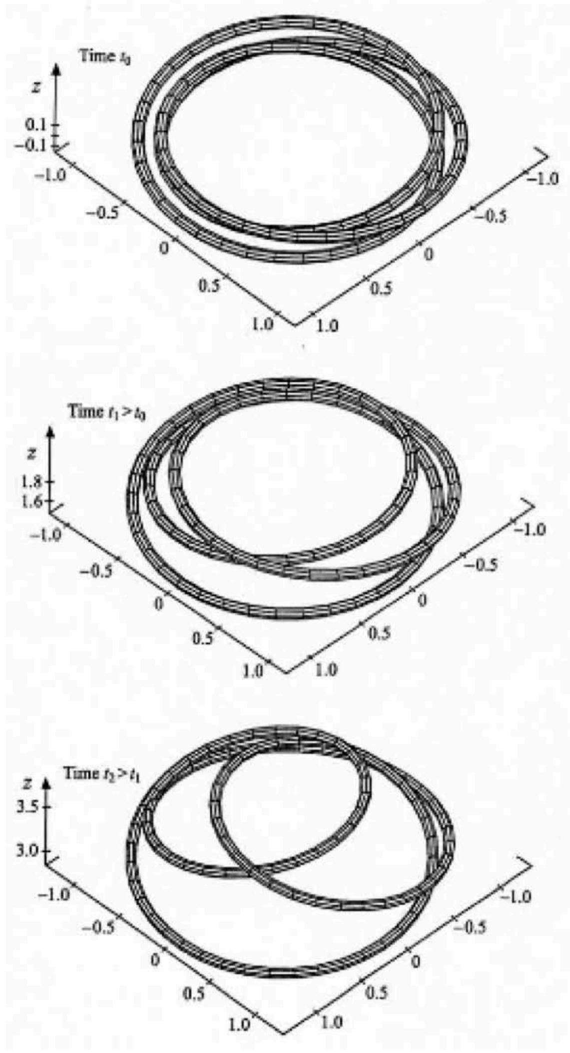
- **Theorem** **(Ricca 1995).** **For any given** $w = q/p$, $\mathcal{T}_{p,q}$ **is linearly stable** ***iff*** $w > 1$.

15-20 diameters



$\mathcal{T}_{2,3}$ $(w > 1)$

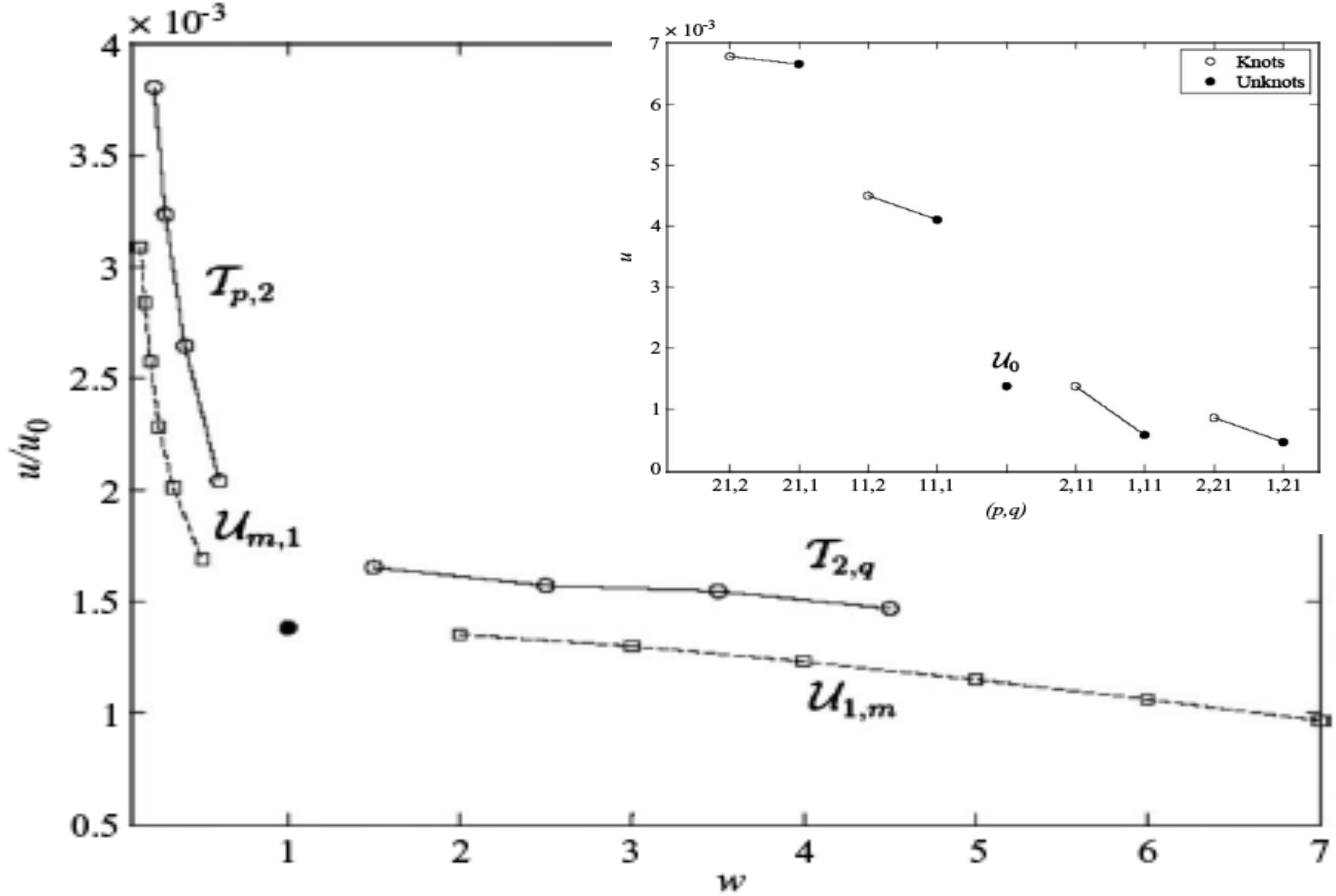
2-3 diameters



$\mathcal{T}_{3,2}$ $(w < 1)$

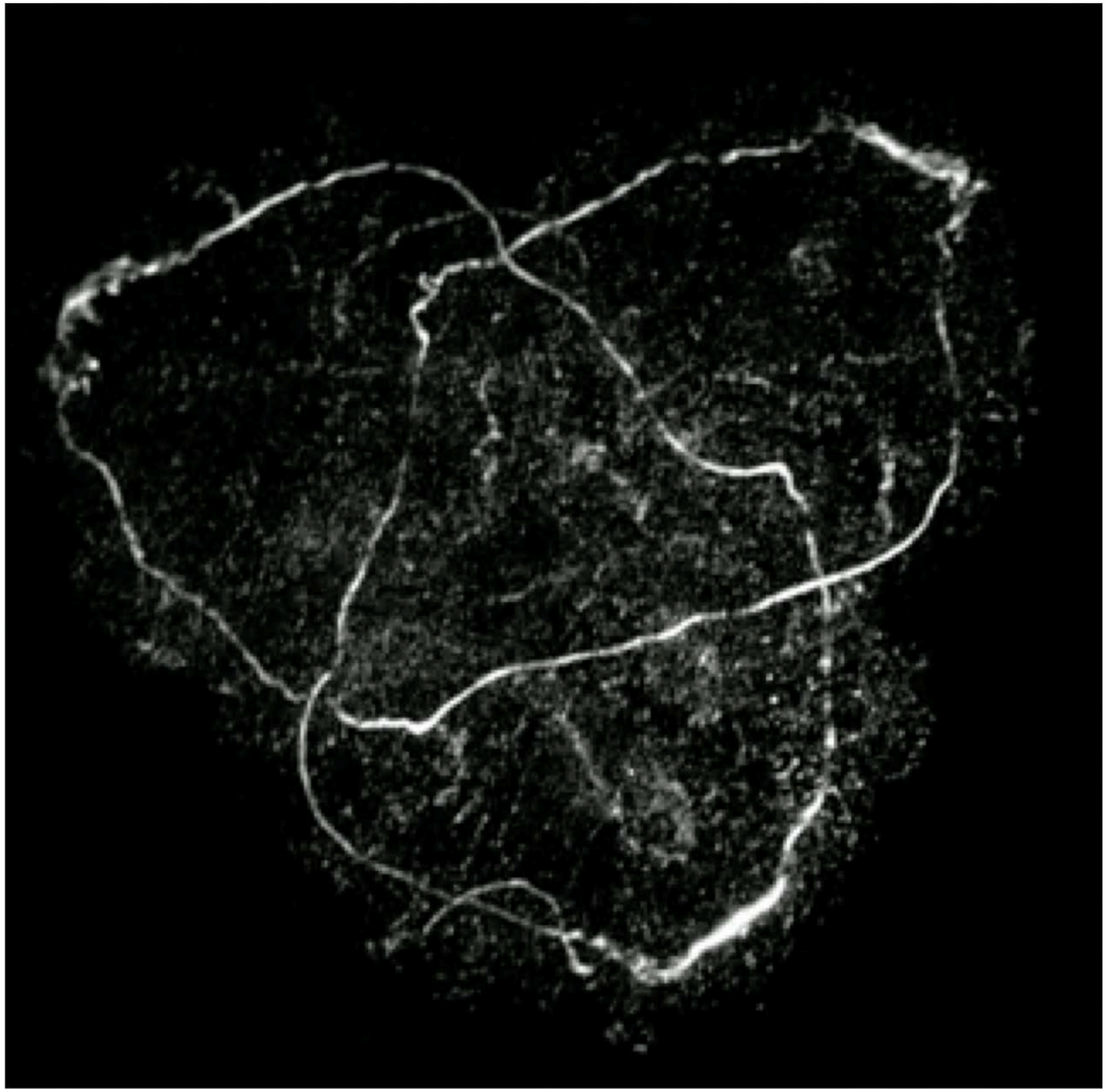
(Ricca et al 1999)

LIA knots $\mathcal{T}_{p,q}$ and unknots $\mathcal{U}_{1,m}$, $\mathcal{U}_{m,1}$



(Maggioni *et al.* 2010)

Creation and dynamics of trefoil vortex knot in water

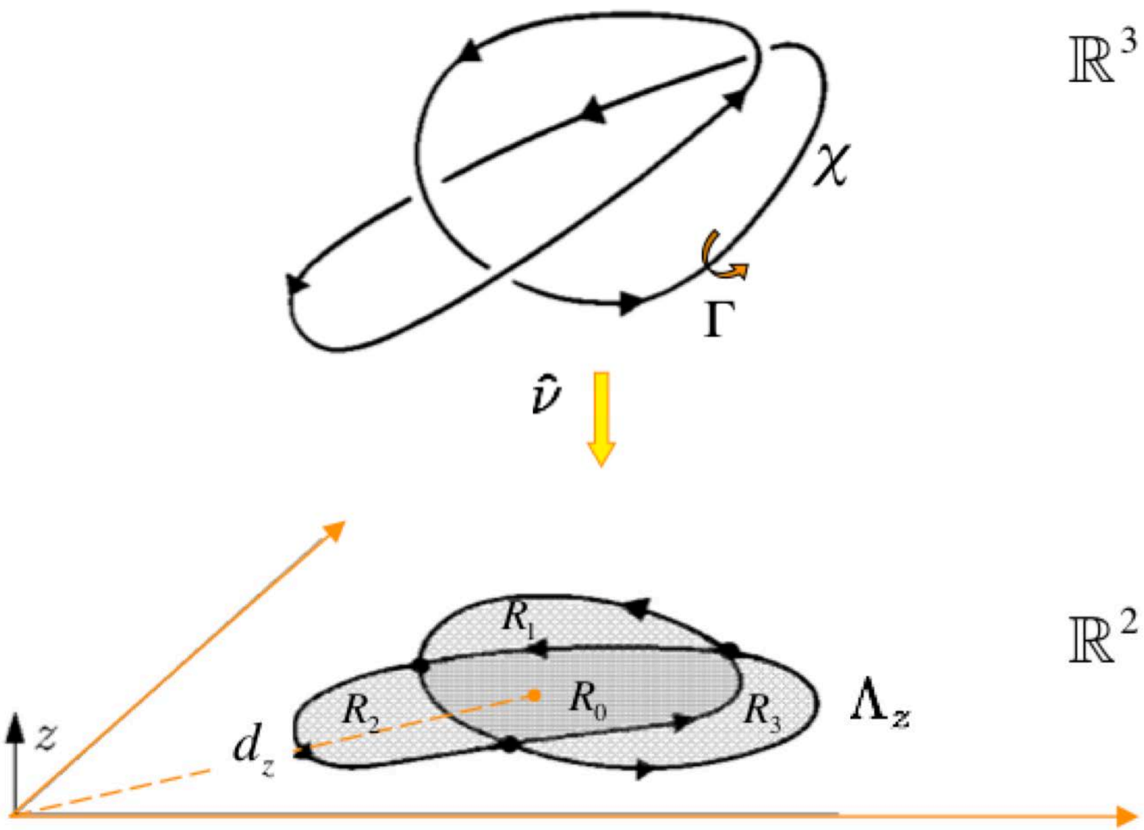


(Klecker & Irvine 2013)

Linear and angular momentum by signed area information

Let χ *denote the centreline of a vortex filament with* $\boldsymbol{\omega} = \omega_0 \hat{\boldsymbol{t}}$ *(* $\omega_0 = \text{cst.}$ *).*

- **Theorem (Ricca 2008).** *The components of linear and angular momentum of a vortex filament of circulation* Γ *, given respectively by* $\boldsymbol{P} = (P_x, P_y, P_z)$ *and* $\boldsymbol{M} = (M_x, M_y, M_z)$*, can be expressed in terms of signed areas of the projected graph regions:*



$$\mathbf{P} \begin{cases} P_x = \Gamma \mathcal{A}(\Lambda_x) \\ P_y = \ldots \\ P_z = \ldots \end{cases},$$

$$\mathbf{M} \begin{cases} M_x = \dfrac{2}{3}\Gamma d_x \mathcal{A}(\Lambda_x) \\ M_y = \ldots \\ M_z = \ldots \end{cases}.$$

Proof.

Let χ ***denote the vortex axis:*** $\boldsymbol{\omega} = \omega_0 \hat{\boldsymbol{t}}$ ***(*** $\omega_0 = \text{cst.}$ ***) , and*** $\Gamma = \int_{\Sigma} \omega \cdot d^2\mathbf{X}$ ***;***

- the linear momentum is given by

$$\boldsymbol{P} = \frac{1}{2}\int_{V(\omega)} \boldsymbol{X} \times \boldsymbol{\omega} \, \mathrm{d}^3\boldsymbol{X} = \frac{1}{2}\Gamma \oint_{L(\chi)} \boldsymbol{X} \times \boldsymbol{X}' \, \mathrm{d}s = \Gamma \oint_{A(\chi)} d^2\boldsymbol{X}$$

- the angular momentum is given by

$$\boldsymbol{M} = \frac{1}{3}\int_{V(\omega)} \boldsymbol{X} \times (\boldsymbol{X} \times \boldsymbol{\omega}) \mathrm{d}^3\boldsymbol{X} = \frac{1}{3}\Gamma \oint_{L(\chi)} \boldsymbol{X} \times (\boldsymbol{X} \times \boldsymbol{X}') \mathrm{d}s = \frac{2}{3}\Gamma \oint_{A(\chi)} d^2\boldsymbol{X}$$

since $\boldsymbol{X}' = \hat{\boldsymbol{t}} = \dfrac{d\boldsymbol{X}}{ds}$***, we have*** $\boldsymbol{X}' \, \mathrm{d}s = \mathrm{d}\boldsymbol{X}$ ***and*** $\oint_{L(\chi)} \mathbf{X} \times d\mathbf{X} = 2\int_{A(\chi)} d^2\mathbf{X}$ ***.***

How to compute $A(\chi)$***?***

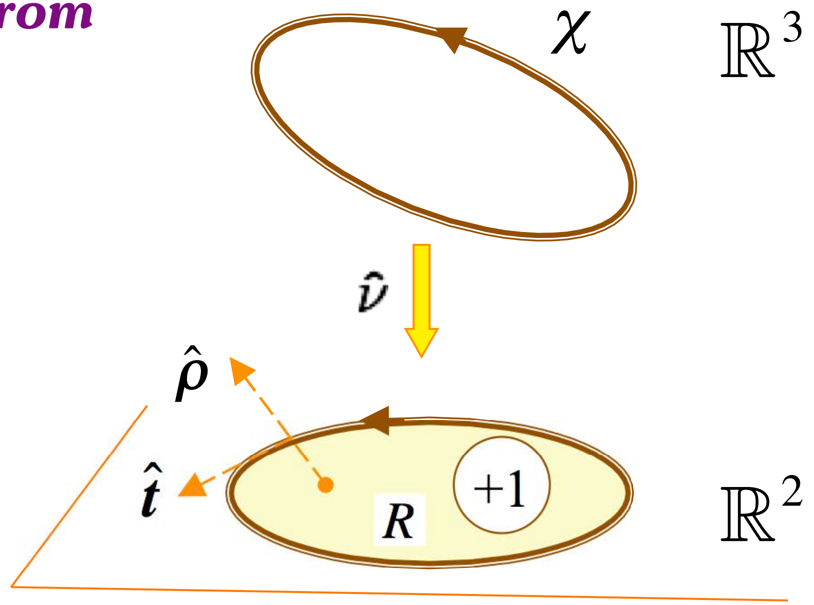
For individual components, we have

$$\Lambda = \hat{v}(\chi) : \ \mathbb{R}^3 \to \mathbb{R}^2 \qquad \begin{cases} P_x : \ \Lambda_x \ \ \textbf{and} \ \ \mathcal{A}_x = \mathcal{A}(\Lambda_x) \\ P_y : \ \Lambda_y \ \ \textbf{and} \ \ \mathcal{A}_y = \mathcal{A}(\Lambda_y) \\ P_z : \ \Lambda_z \ \ \textbf{and} \ \ \mathcal{A}_z = \mathcal{A}(\Lambda_z) \ , \end{cases}$$

where Λ_x ***denotes the standard graph projected along the*** x***-axis and*** $\mathcal{A}_x$ ***the signed area of*** Λ_x ***. Similarly for the*** y***- and*** z***-axis and*** M_x ***,*** M_y ***,*** M_z ***.***

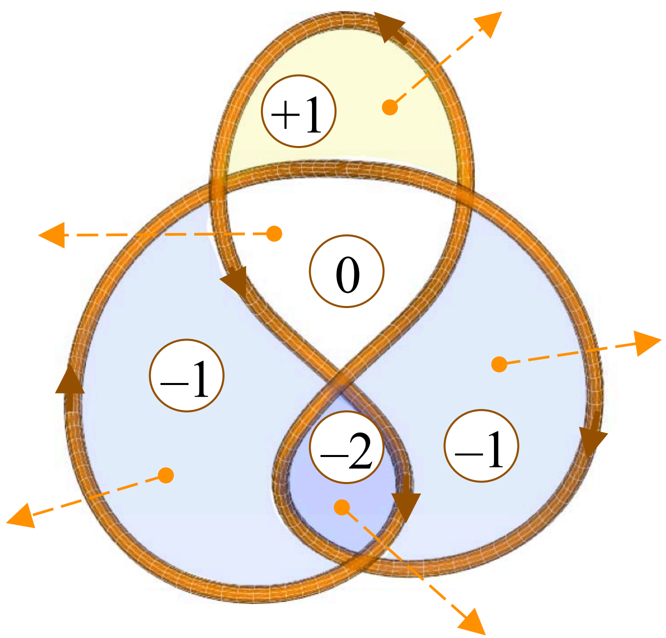
Computation of signed area of a planar graph

To each sub-region R_j of Λ we assign an index $\mathcal{I}_j$ that weights the relative contribution from the circulation of neighbouring strands:



$$\mathcal{I}_j = \mathcal{I}(R_j) = \sum_{r \in \{\rho \cap \Lambda\}} \epsilon_r$$

where $\epsilon_r = \pm 1$ according to the r.h. sign convention of the reference $\{\hat{\rho}, \hat{\mathbf{t}}\}$.



The signed area $\mathcal{A}$ of Λ is thus given by

$$\mathcal{A}(\Lambda) = \sum_j \mathcal{I}_j A_j$$

where $A_j = A(R_j)$ is the standard area of R_j.

Thus, by the standard definitions of momenta and the observations above we have:

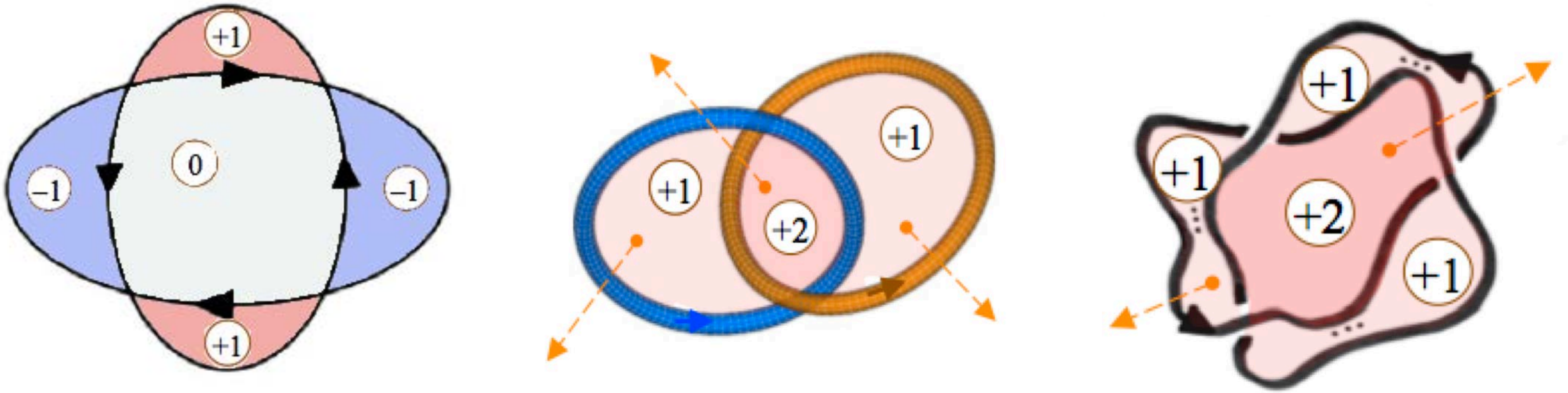
$$\boldsymbol{P} = \frac{1}{2}\Gamma\oint_{L(\chi)} \boldsymbol{X}\times\boldsymbol{X}'\,\mathrm{d}s = \Gamma\int_{A(\chi)} d^2\boldsymbol{X} = \Gamma\vec{\mathcal{A}}$$

$$\boldsymbol{M} = \frac{1}{3}\Gamma\oint_{L(\chi)} \boldsymbol{X}\times(\boldsymbol{X}\times\boldsymbol{X}')\,\mathrm{d}s = \frac{2}{3}\Gamma\int_{A(\chi)} \boldsymbol{X}\times d^2\boldsymbol{X} = \frac{2}{3}\Gamma d\vec{\mathcal{A}}$$

where $\vec{\mathcal{A}} = (\mathcal{A}_x, \mathcal{A}_y, \mathcal{A}_z)$, $d\vec{\mathcal{A}} = (d_x\mathcal{A}_x, d_y\mathcal{A}_y, d_z\mathcal{A}_z)$ *and* $\mathcal{A}_x = \mathcal{A}(\Lambda_x)$ *the signed areas of the projected graphs.*

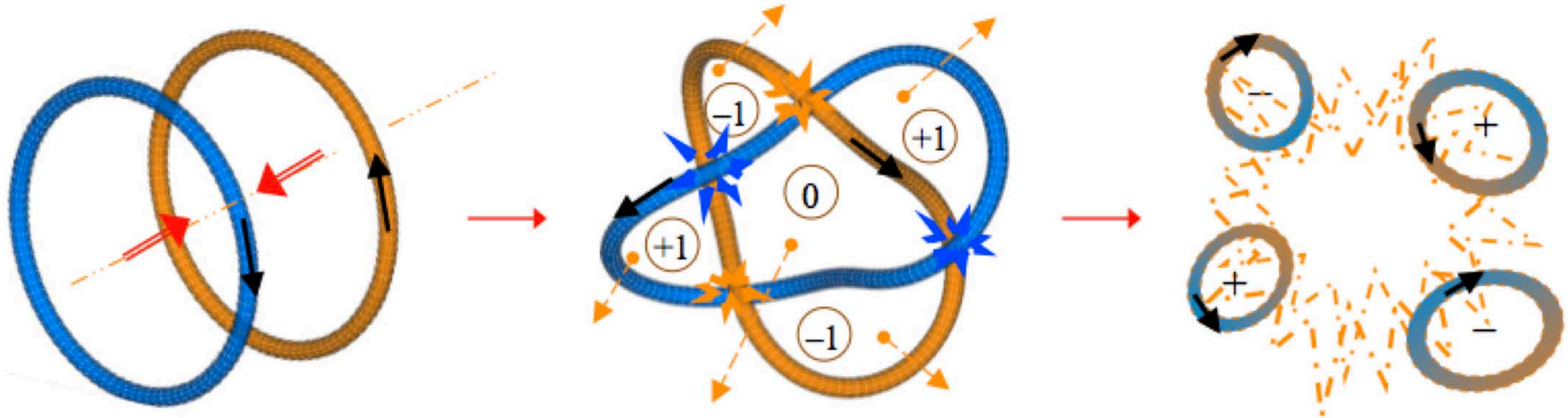
■

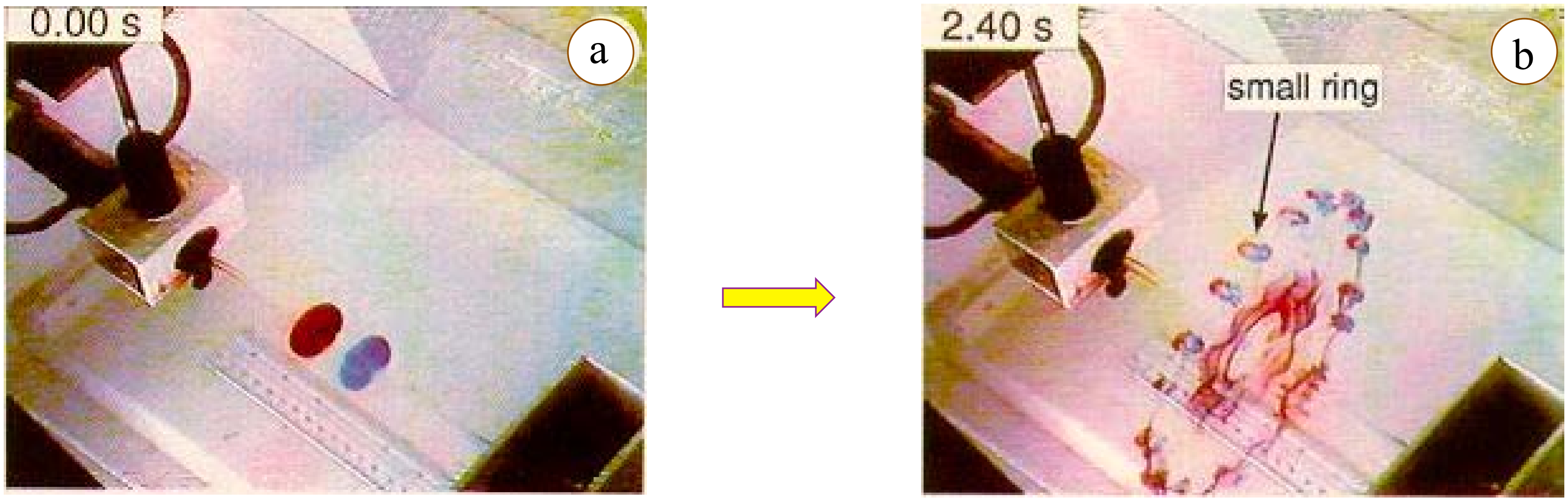
- *Computation of signed area of interacting vortices ($\Gamma=1$)*



- **Corollary.** *The components of linear and angular momentum of a vortex tangle $\mathcal{T}$ can be computed in terms of signed areas of the projected graph regions.*

Head-on collision of vortex rings





(Lim & Nickels 1992)