Cascade process of linked quantum vortex loops

Simone Zuccher[1] and Renzo L. Ricca[2]

[1] *Department of Computer Science, University of Verona, Italy*
[2] *Department of Mathematics & Applications, University of Milano-Bicocca, Italy*

Summary

Here we present for the first time a detailed study of the cascade process of two linked quantum vortex loops under the Gross-Pitaevskii equation. After a series of successive reconnections the vortex system decays to three distinct loops. Kinetic helicity as well as various geometric and topological properties associated with helicity are examined throughout the reconnection events, providing a complete structural analysis of the whole process.

1 Quantum vortex evolution under the Gross-Pitaevskii equation

Quantum vortex dynamics is governed by the Gross-Pitaevskii equation (GPE)

$$\frac{\partial\psi}{\partial t} = \frac{\mathrm{i}}{2}\boldsymbol{\nabla}^2\psi + \frac{\mathrm{i}}{2}(1-|\psi|^2)\psi \ , \tag{1}$$

with background density $\rho_b = 1$. It is well-known that under the Madelung transformation $\psi = \sqrt{\rho}\exp(\mathrm{i}\theta)$ eq. (1) can be written in terms of a continuity equation and a momentum equation of an ideal fluid with density $\rho = |\psi|^2$ and velocity $\boldsymbol{u} = \boldsymbol{\nabla}\theta$. Defects in the wave function ψ are infinitesimally thin vortices of constant circulation $\Gamma = \oint \boldsymbol{u}\cdot \mathrm{d}\boldsymbol{s} = 2\pi$ and healing length $\xi = 1$. An important physical quantity is kinetic helicity, given by

$$H = \int \boldsymbol{u}\cdot\boldsymbol{\omega}\ \mathrm{d}^3\boldsymbol{x} \ , \tag{2}$$

that in ideal conditions is known to be conserved. In the case of two distinct vortex loops, helicity can be written in terms of Gauss' linking number Lk and Călugăreanu-White's self-linking number SL [1, 2]. Since for a 2-component link $Lk_{12} = Lk_{21}$ and vortex circulations are equal, we have

$$H = \Gamma^2\left(2Lk_{12} + SL_1 + SL_2\right) \ , \tag{3}$$

where $SL_i = Wr_i + T_i + N_i$ $(i = 1, 2)$ admits decomposition in terms of writhe Wr_i, total torsion T_i and intrinsic twist N_i of the i-th vortex centerline. Helicity can therefore be computed by two independent methods; one based on the velocity-vorticity distribution by using eq. (2), and the other by extracting geometric and topological information by using eq. (3). In recent numerical simulations [3] we implemented the computation of all these quantities to study the reconnection of two vortex loops. Details of the numerical code and of the procedure adopted for the extraction of the vortex centerline can be found there. In particular we showed that the intrinsic twist can be computed by identifying the reference ribbon defined on the vortex centerline with the phase associated with the wave function ψ, a prescription that provides reliable information for the computation of geometric and dynamical properties involved during reconnection events as well. That exercise served as a test benchmark for the present investigation.

2 Cascade process of two linked vortex loops

Here we present preliminary results on the evolution of two linked, vortex loops that, through interaction and successive reconnections, untie and generate a system of unlinked, unknotted loops.

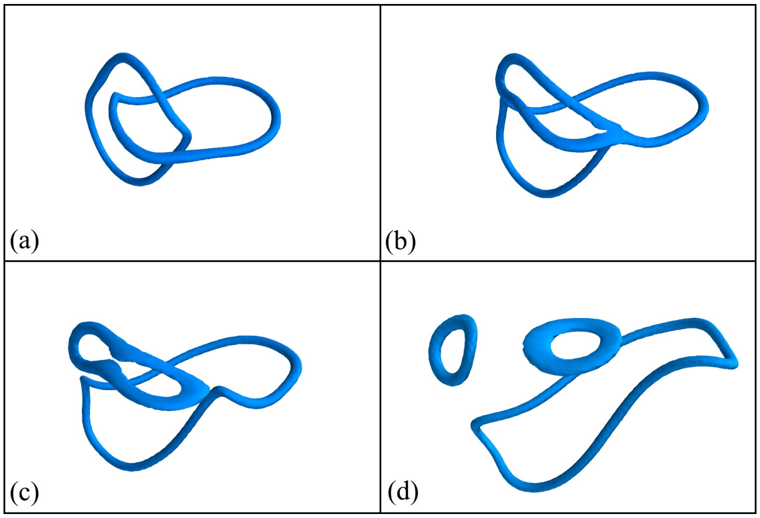

Fig. 1: Time evolution of two linked vortex loops under GPE. (a) The vortex system undergoes a first reconnection to form (b) a large, folded loop. (c) After a subsequent reconnection two secondary unlinked, vortex loops are produced, and after a third reconnection (d) the system decays to form three distinct loops. Tubes visualize constant density surfaces.

The whole process is shown in Figure 1. After a first reconnection the two loops untie to form a large folded loop (Figure 1b), that quickly reconnects to produce a secondary structure that after a third reconnection generates two smaller loops. This cascade process reveals interesting features and some details are similar to analogous experiments done by using a three-dimensional vortex-in-cell code in a nearly inviscid context [4].

A detailed analysis of the geometric, topological and dynamical properties associated with helicity is presented. Accurate computations of all the quantities across scales and during each reconnection event are discussed and compared with helicity computation based on eq. (2).

References

[1] Moffatt, H.K. (1969) The degree of knottedness of tangled vortex lines *J.Fluid Mech.* **35**, 117–129.

[2] Moffatt, H.K. & Ricca, R.L. (1992) Helicity and the Călugareănu invariant. *Proc. R. Soc. Lond. A* **439**, 411–429.

[3] Zuccher, S. & Ricca, R.L. (2015) Helicity conservation under quantum reconnection of vortex rings. *Phys. Rev. E* **92**, 061001–5.

[4] Aref, H. & Zawadzki, I. (1991) Linking of vortex rings. *Nature* **354**, 50–53.