Rattleback: a prototype of chiral dynamics

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Summary

A rattleback is a rigid body whose ellipsoid of inertia is skewed with respect to the principal axes of the contact surface. The chirality of this toy, the breaking of mirror symmetry as manifested by rotational preference, gives us a hint to delineate the origin of chiral dynamics in various systems ranging from classical fluids/plasmas to quantum regimes. From the rattleback rigid-body equations of motion one can extract a prototypical rattleback system composed of three first order coupled differential equations [1], which, in the non-dissipative idealization, has two first integrals, energy and an intriguing function. It is the further study of this system that is the purpose of the present work; we devise a noncanonical three-dimensional Hamiltonian system. The underlying Lie algebra is of Bianchi Type VI, which appears for the first time in a physical example. Its Casimir, whose existence makes the system noncanonical, is the intriguing first integral. The chirality of the rattleback motion is caused by the geometric skewness of the leaf.

1 Model equations and Hamiltonian formalism

The equations of the prototypical rattleback system (called here PRS) are as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} P \\ R \\ S \end{pmatrix} = \begin{pmatrix} R \\ \lambda P \\ 0 \end{pmatrix} \times \begin{pmatrix} P \\ R \\ S \end{pmatrix} = \begin{pmatrix} \lambda PS \\ -RS \\ R^2-\lambda P^2 \end{pmatrix}. \tag{1}$$

Compared with equation (5.5) of [1], we have adopted a more felicitous notation where P, R, S stand for the *pitching*, *rolling*, and *spinning* modes of the motion. The quantity λ is a positive parameter that encodes the *aspect ratio* of the rattleback.

Despite the fact that the phase space is odd-dimensional, it is possible, and useful, to cast the equations (1) in a Hamiltonian form; the system becomes noncanonical and the [2], In terms of the coordinates $\boldsymbol{X} = (P\ R\ S)^{\mathrm{T}} \in \Omega \subset \mathbf{R}^3$, define a Poisson matrix

$$J = \begin{pmatrix} 0 & 0 & \lambda P \\ 0 & 0 & -R \\ -\lambda P & R & 0 \end{pmatrix}, \tag{2}$$

and denote by $\langle\ ,\ \rangle_\Omega$ the standard inner product on the phase space Ω. If we take the *Hamiltonian*

$$H = \frac{1}{2}\langle \boldsymbol{X}, \boldsymbol{X}\rangle_\Omega = \frac{1}{2}\left(P^2+R^2+S^2\right), \tag{3}$$

then (1) is cast into the following Hamilton form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{X} = J\partial_{\boldsymbol{X}} H = \{\boldsymbol{X}, H\}_J, \tag{4}$$

where in the second equality the Poisson bracket $\{F, G\}_J = \langle \partial_{\boldsymbol{X}} F, J\partial_{\boldsymbol{X}} G\rangle_\Omega$ makes $C^\infty(\Omega)$ into a Poisson algebra, a Lie algebra realization on functions, since it is bilinear, antisymmetric, and can be shown to satisfy the Jacobi identity.

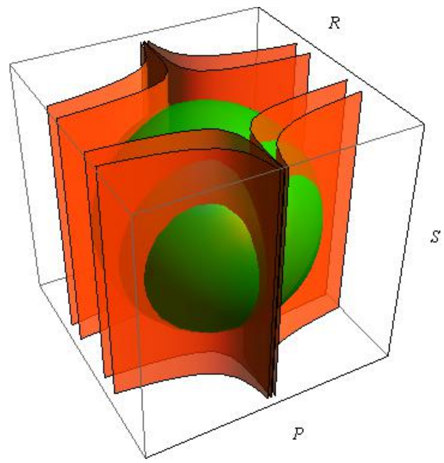

Fig. 1: Orbits are the intersections of an energy level $H = \frac{1}{2}(P^2 + R^2 + S^2) = \text{const.}$ (green sphere) and a Casimir surface $C = PR^\lambda = \text{const.}$ (red curved sheet); the leaves $C = 1, 0.1, -0.1, -1$ are shown. The aspect-ratio parameter is taken to be $\lambda = 4$.

The degeneracy of J yields a *Casimir invariant*

$$C = PR^\lambda, \tag{5}$$

characterized by the property $\{C, G\}_J = 0 \ \forall G \in C^\infty(\Omega)$. Since

$$\det(zI - J) = z(z^2 + \lambda^2 P^2 + R^2),$$

we have $\text{rank}J = 2$ except along the singular set $\lambda^2 P^2 + R^2 = 0$, i.e. $P = R = 0$, where $\text{rank}J$ drops to 0. In this paper we study the dynamics in the regime $\lambda^2 P^2 + R^2 > 0$; for example it suffices to take the phase space Ω to be an open set in $\mathbf{R}^3$ on which $|C| = |PR^\lambda|$ is bounded away from 0.

2 Skewed space and chirality

Thanks to the Hamiltonian structure, the system (1) is integrable. Examination of the intersection of the Hamiltonian with the Casimir reveals that every orbit is periodic (cf. Fig. 1).

The curious guiding idea of this paper is that we can formulate the dynamics as if a symmetric body were moving in an asymmetric space—recall the formalism of the previous section, where the Hamiltonian is symmetric but there is something asymmetric in the phase space of the Bianchi type $\text{VI}_{h\neq-1}$ Lie-Poisson algebra. The chirality, then, comes from the skewness of the Casimir leaves.

Philosophically, it is interesting to think of the interchangeability of a Hamiltonian and a Casimir as a duality between matter and space. A complex material system may be made simple by transferring complexity to the geometry of space; conversely, a skewed space may be made flat by skewing the matter or the energy. The list of Bianchi algebras provides us with plenty of transfer possibilities.

References

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