Variational Principles for non-Barotropic Magnetohydrodynamics and Local Topological Conservation Laws

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Summary

Variational principles for magnetohydrodynamics (MHD) were introduced by previous authors both in Lagrangian and Eulerian form. In this talk we introduce simpler Eulerian variational principles which is appropriate for some magnetic field topologies from which all the relevant equations of non-barotropic MHD can be derived. The variational principle is given in terms of five independent functions for non-stationary barotropic flows. This is less then the eight variables which appear in the standard equations of barotropic MHD which are the magnetic field $\vec{B}$ the velocity field $\vec{v}$, the entropy s and the density ρ. The functions which are needed in addition to the entropy and density are two co-moving surfaces χ and η (Euler potentials) and a multiple valued Bernoulli type function ν. We will show that the discontinuity of ν is a conserved quantity and contains information about the cross helicity per unit of magnetic flux.

1 Barotropic MHD

Variational principles for MHD were introduced by previous authors both in Lagrangian and Eulerian form. Sturrock [1] has discussed in his book a Lagrangian variational formalism for MHD. Vladimirov and Moffatt [2] in a series of papers have discussed an Eulerian variational principle for incompressible MHD. However, their variational principle contained three more functions in addition to the seven variables which appear in the standard equations of incompressible MHD which are the magnetic field $\vec{B}$ the velocity field $\vec{v}$ and the pressure P. Kats [3] has generalized Moffatt's work for compressible non barotropic flows but without reducing the number of functions and the computational load. Sakurai [4] has introduced a two function Eulerian variational principle for force-free MHD and used it as a basis of a numerical scheme, his method is discussed in a book by Sturrock [1]. Yahalom & Lynden-Bell [5] combined the Lagrangian of Sturrock [1] with the Lagrangian of Sakurai [4] to obtain an **Eulerian** Lagrangian principle for barotropic MHD which will depend on only six functions. The variational derivative of this Lagrangian produced all the equations needed to describe barotropic MHD without any additional constraints. The equations obtained resembled the equations of Frenkel, Levich & Stilman [6]. Yahalom [7] have shown that for the barotropic case four functions will suffice. Moreover, it was shown that the cuts of some of those functions [8] are topological local conserved quantities. It was also shown [5] that the cross helicity conservation law can be derived via the Noether theorem using a method similar to the method used to derive the conservation of helicity in non magnetic fluids [9].

2 Non-Barotropic MHD

Variational principles of non barotropic MHD can be found in the work of Bekenstein & Oron [10] in terms of 15 functions and V.A. Kats [3] in terms of 20 functions. Moreover, A. V. Kats in a remarkable paper [11] (section IV,E) has shown that there is a large symmetry group (gauge freedom) associated with the choice of those functions, this implies that the number of degrees of freedom can be reduced. Morrison [12] has suggested a Hamiltonian approach but this also depends on 8 canonical variables (see table 2 [12]). Here we will show that only five functions will suffice to describe non barotropic MHD in the case that we enforce a Sakurai [4] representation for the magnetic field. The functions which are needed in addition to the entropy and density are two co-moving surfaces χ and η (Euler potentials) and a multiple valued Bernoulli type function ν [13, 14].

We will show that the discontinuity of ν is a conserved quantity and contains information about the cross helicity per unit of magnetic flux. (See [8] for the analogous barotropic case)

We anticipate applications of this study both to linear and non-linear stability analysis of known non barotropic MHD configurations [15, 16, 17, 18] and for designing efficient numerical schemes for integrating the equations of fluid dynamics and MHD [19, 20, 21, 22]. Another possible application is connected to obtaining new analytic solutions in terms of the variational variables [23].

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