Optimal dynamical systems of Navier-Stokes equations based on generalized helical-wave bases and the fundamental elements of turbulence

C. J. Wu[1], N. F. Peng[1,2], H. Guan[3], Z. J. Wei[2]

[1] *State Key Laboratory of Structural Analysis for Industrial Equipment, School of Aeronautics and Astronautics, Dalian University of Technology, Dalian 116024, China*
[2] *State Key Laboratory for Turbulence and Complex Systems, Center for Applied Physics and Technology, College of Engineering, Peking University, Beijing 100871, China*
[3] *College of Meteorology and Oceanography, PLA University of Science and Technology, Nanjing 211101, China*

Summary

In this paper, we present the theory of constructing optimal generalized helical-wave coupling dynamical systems. Applying the helical-wave decomposition method to Navier-Stokes equations, we derive a pair of coupling dynamical systems. With the method of multi-scale global optimization based on coarse graining analysis, a set of global optimal generalized helical-wave bases is obtained. On the one hand, optimal generalized helical-wave bases retain the good properties of classical helical-wave bases; on the other hand, they are optimal for the dynamical systems of Navier-Stokes equations, and suitable for complex physical and geometric boundary conditions. Then we find that the optimal generalized helical-wave vortexes fitted by a finite number of optimal generalized helical-wave bases can be used as the fundamental elements of turbulence, and have important significance for studying physical properties of complex flows and turbulent vortex structures in a deeper level.

1 The fundamental element of turbulence and its properties

In the researches of turbulent physics, we believe that there exist the fundamental elements of turbulence (Wu & Zhao [1], 2000), which should have the following properties.

(1) In dynamical aspect, they must deeply reflect the typical dynamical properties of turbulence, such as the process of shear and expansion, fluctuation, and chaos.

(2) In physical space, they must be concentrated, and decay fast enough at the far field.

(3) At the time revolution aspect, they must have the property of slow changing, i.e., the evolution of fundamental elements is much slower than the fluctuations of flow field's evolution, so the nonlinear interaction among these fundamental elements' slow changing causes fast variations of turbulent flows.

In our previous researches, it has been proved that the vortex element is not a good candidate of the fundamental element of turbulence, while the helical-wave base based on the method of helical-wave decomposition [2, 3] (HWD) is a good one. However, classical helical-wave bases have three kinds of drawbacks [1]. In this way, we find a new kinds of fundamental elements of turbulence called optimal generalized helical-wave bases.

2 Optimal generalized helical-wave coupling dynamical systems of Navier-Stokes equations

As for the general three-dimensional incompressible viscous flow, non-dimensional Navier-Stokes equations [4], continuity equations, initial and boundary conditions after HWD are as follows.

$$\begin{cases} \dot{u}_i^+ + \dot{u}_i^- + u_j^+ u_{i,j}^+ + u_j^+ u_{i,j}^- + u_j^- u_{i,j}^+ + u_j^- u_{i,j}^- + p_{,i} - \dfrac{1}{Re}\left(u_{i,jj}^+ + u_{i,jj}^-\right) = 0 \\ u_{i,i}^+ = u_{i,i}^- = 0 \\ u_i^+\big|_{t=0} = u_{0i}^+,\ u_i^-\big|_{t=0} = u_{0i}^- \\ \left[\alpha\left(u_i^+ + u_i^-\right) + \beta\left(u_{i,j}^+ + u_{i,j}^-\right)n_j\right]_{\partial\Omega} = \alpha\left(u_{Bi}^+ + u_{Bi}^-\right) + \beta\left(u_{Bi,j}^+ + u_{Bi,j}^-\right)n_j \end{cases} \tag{1}$$

For low-dimensional model of Eq.(1), the problem can be limited in N-dimensional Hilbert space, so the function space $\mathcal{B}_N^{\mathrm{HWD}}$ spanned by generalized helical-wave bases $\xi_{\hat{k}i}^{\pm}$ is:

$$\begin{aligned} \mathcal{B}_N^{\mathrm{HWD}} = \Bigg\{ [\xi_{\hat{k}i}^{\pm}]_{\hat{k}=1}^{N} \Bigg|\ \xi_{\hat{k}i}^{\pm} \in \mathcal{H}^N(\Omega),\ \int_\Omega \xi_{\hat{k}i}^{\pm}\xi_{li}^{\pm}\,\mathrm{d}\Omega = \delta_{\hat{k}l},\ \int_\Omega \xi_{\hat{k}i}^{+}\xi_{li}^{-}\,\mathrm{d}\Omega = 0, \\ \text{and } \xi_{\hat{k}i}^{\pm} \text{ is twice differentiable; } \hat{k} \neq l,\ \varepsilon_{ijk}\xi_{\hat{k}i}^{\pm}\xi_{lk,j}^{\pm} = 0 \Bigg\} \end{aligned} \tag{2}$$

And the low-dimensional approximate expressions of $u_i^{\pm}$ in $\mathcal{B}_N^{\mathrm{HWD}}$ are:

$$u_i^{\pm} \approx a_{\hat{k}}^{\pm}\xi_{\hat{k}i}^{\pm} \tag{3}$$

Where the truncated order is N, which is omitted.

Under the theoretical framework of constructing optimal dynamical systems based on weighted residual [5], the objective functional and the generalized objective functional without constraint conditions can be built. Then by the variational operation, we eventually obtain the optimal generalized helical-wave coupling dynamical systems of Navier-Stokes equations as follows.

(1) The ODEs of $a_{\hat{k}}^{\pm}$ (optimal generalized helical-wave coupling dynamical systems).

$$\begin{aligned} &\dot{a}_{\hat{k}}^{+} + B_{\hat{k}lm}^{+++}a_l^+a_m^+ + B_{\hat{k}lm}^{++-}a_l^+a_m^- + B_{\hat{k}lm}^{+-+}a_l^-a_m^+ \\ &+ B_{\hat{k}lm}^{+--}a_l^-a_m^- + \frac{1}{Re}\left(C_{\hat{k}l}^{++}a_l^+ + C_{\hat{k}l}^{+-}a_l^-\right) = 0 \end{aligned} \tag{4a}$$

$$\begin{aligned} &\dot{a}_{\hat{k}}^{-} + B_{\hat{k}lm}^{---}a_l^-a_m^- + B_{\hat{k}lm}^{--+}a_l^-a_m^+ + B_{\hat{k}lm}^{-+-}a_l^+a_m^- \\ &+ B_{\hat{k}lm}^{-++}a_l^+a_m^+ + \frac{1}{Re}\left(C_{\hat{k}l}^{--}a_l^- + C_{\hat{k}l}^{-+}a_l^+\right) = 0 \end{aligned} \tag{4b}$$

(2) The ODEs of $\lambda_{\hat{k}}^{\pm}$.

$$\begin{aligned} \dot{\lambda}_{\hat{k}}^{+} = \ &\left(B_{l\hat{k}m}^{+++} + B_{lm\hat{k}}^{+++}\right)\tilde{\lambda}_l^+a_m^+ + \left(B_{l\hat{k}m}^{++-} + B_{lm\hat{k}}^{+-+}\right)\tilde{\lambda}_l^+a_m^- + \\ &\left(B_{l\hat{k}m}^{-++} + B_{lm\hat{k}}^{-++}\right)\tilde{\lambda}_l^-a_m^+ + \left(B_{l\hat{k}m}^{-+-} + B_{lm\hat{k}}^{--+}\right)\tilde{\lambda}_l^-a_m^- + \\ &\frac{1}{Re}\left(C_{\hat{k}l}^{++}\tilde{\lambda}_i^+ + C_{\hat{k}l}^{+-}\tilde{\lambda}_l^-\right) \end{aligned} \tag{5a}$$

$$\begin{aligned} \dot{\lambda}_{\hat{k}}^{-} = \ &\left(B_{l\hat{k}m}^{---} + B_{lm\hat{k}}^{---}\right)\tilde{\lambda}_l^-a_m^- + \left(B_{l\hat{k}m}^{--+} + B_{lm\hat{k}}^{-+-}\right)\tilde{\lambda}_l^-a_m^+ + \\ &\left(B_{l\hat{k}m}^{+--} + B_{lm\hat{k}}^{+--}\right)\tilde{\lambda}_l^+a_m^- + \left(B_{l\hat{k}m}^{+-+} + B_{lm\hat{k}}^{++-}\right)\tilde{\lambda}_l^+a_m^+ + \\ &\frac{1}{Re}\left(C_{\hat{k}l}^{--}\tilde{\lambda}_i^+ + C_{\hat{k}l}^{-+}\tilde{\lambda}_l^+\right) \end{aligned} \tag{5b}$$

(3) The equations of the gradient of the generalized objective functional.

In the domain Ω:

$$
\begin{aligned}
&-\tilde{\lambda}_{\hat{k}}^{+}(0)\, u_{0i}^{+} + P_{\hat{k}lm}^{+++}\xi_{lj}^{+}\xi_{mi,j}^{+} + P_{ml\hat{k}}^{+++}\left(\xi_{mj}^{+}\xi_{lj,i}^{+} - \xi_{lj}^{+}\xi_{mi,j}^{+}\right) + P_{\hat{k}lm}^{++-}\xi_{lj}^{+}\xi_{mi,j}^{-} \\
&+ P_{ml\hat{k}}^{-++}\left(\xi_{mj}^{-}\xi_{lj,i}^{+} - \xi_{lj}^{+}\xi_{mi,j}^{-}\right) + P_{\hat{k}lm}^{+-+}\xi_{lj}^{-}\xi_{mi,j}^{+} + P_{ml\hat{k}}^{+-+}\left(\xi_{mj}^{+}\xi_{lj,i}^{-} - \xi_{lj}^{-}\xi_{mi,j}^{+}\right) \\
&+ P_{\hat{k}lm}^{+--}\xi_{lj}^{-}\xi_{mi,j}^{-} + P_{ml\hat{k}}^{--+}\left(\xi_{mj}^{-}\xi_{lj,i}^{-} - \xi_{lj}^{-}\xi_{mi,j}^{-}\right) - \frac{1}{Re}\left(S_{\hat{k}l}^{++} + S_{l\hat{k}}^{++}\right)\xi_{li,jj}^{+} \\
&- \frac{1}{Re}\left(S_{\hat{k}l}^{+-} + S_{l\hat{k}}^{-+}\right)\xi_{li,jj}^{-} + 4\mu^{+}\left(E_{ll}^{++} - 1\right)\xi_{\hat{k}i}^{+} + 2\mu^{+}\left(E_{\hat{k}l}^{++}\xi_{li}^{+}\right)_{\hat{k}\neq l} \\
&+ 2\nu^{+}\left(\varepsilon_{ijk}F_{\hat{k}l}^{++}\xi_{\hat{k}k,j}^{+}\right)_{\hat{k}\neq l} = 0
\end{aligned}
\tag{6a}
$$

$$
\begin{aligned}
&-\tilde{\lambda}_{\hat{k}}^{-}(0)\, u_{0i}^{-} + P_{\hat{k}lm}^{---}\xi_{lj}^{-}\xi_{mi,j}^{-} + P_{ml\hat{k}}^{---}\left(\xi_{mj}^{-}\xi_{lj,i}^{-} - \xi_{lj}^{-}\xi_{mi,j}^{-}\right) + P_{\hat{k}lm}^{--+}\xi_{lj}^{-}\xi_{mi,j}^{+} \\
&+ P_{ml\hat{k}}^{+--}\left(\xi_{mj}^{+}\xi_{lj,i}^{-} - \xi_{lj}^{-}\xi_{mi,j}^{+}\right) + P_{\hat{k}lm}^{-+-}\xi_{lj}^{+}\xi_{mi,j}^{-} + P_{ml\hat{k}}^{-+-}\left(\xi_{mj}^{-}\xi_{lj,i}^{+} - \xi_{lj}^{+}\xi_{mi,j}^{-}\right) \\
&+ P_{\hat{k}lm}^{-++}\xi_{lj}^{+}\xi_{mi,j}^{+} + P_{ml\hat{k}}^{++-}\left(\xi_{mj}^{+}\xi_{lj,i}^{+} - \xi_{lj}^{+}\xi_{mi,j}^{+}\right) - \frac{1}{Re}\left(S_{\hat{k}l}^{--} + S_{l\hat{k}}^{--}\right)\xi_{li,jj}^{-} \\
&- \frac{1}{Re}\left(S_{\hat{k}l}^{-+} + S_{l\hat{k}}^{+-}\right)\xi_{li,jj}^{+} + 4\mu^{-}\left(E_{ll}^{--} - 1\right)\xi_{\hat{k}i}^{-} + 2\mu^{-}\left(E_{\hat{k}l}^{--}\xi_{li}^{-}\right)_{\hat{k}\neq l} \\
&+ 2\nu^{-}\left(\varepsilon_{ijk}F_{\hat{k}l}^{--}\xi_{\hat{k}k,j}^{-}\right)_{\hat{k}\neq l} = 0
\end{aligned}
\tag{6b}
$$

In the boundary, for Dirichlet boundary conditions:

$$
\begin{aligned}
&S_{m\hat{k}}^{++}\xi_{li}^{+}\left(u_{Bj}^{+} + u_{Bj}^{-}\right)n_j + S_{m\hat{k}}^{-+}\xi_{li}^{-}\left(u_{Bj}^{+} + u_{Bj}^{-}\right)n_j \\
&+ \frac{1}{Re}\left(S_{\hat{k}l}^{++} + S_{l\hat{k}}^{++}\right)\xi_{li,j}^{+}n_j + \frac{1}{Re}\left(S_{\hat{k}l}^{+-} + S_{l\hat{k}}^{-+}\right)\xi_{li,j}^{-}n_j = 0
\end{aligned}
\tag{7a}
$$

$$
\begin{aligned}
&S_{m\hat{k}}^{--}\xi_{li}^{-}\left(u_{Bj}^{+} + u_{Bj}^{-}\right)n_j + S_{m\hat{k}}^{+-}\xi_{li}^{+}\left(u_{Bj}^{+} + u_{Bj}^{-}\right)n_j \\
&+ \frac{1}{Re}\left(S_{\hat{k}l}^{--} + S_{l\hat{k}}^{--}\right)\xi_{li,j}^{-}n_j + \frac{1}{Re}\left(S_{\hat{k}l}^{-+} + S_{l\hat{k}}^{+-}\right)\xi_{li,j}^{+}n_j = 0
\end{aligned}
\tag{7b}
$$

For Newman boundary conditions:

$$
\begin{aligned}
&P_{ml\hat{k}}^{+++}\xi_{mi}^{+}\xi_{lj}^{+}n_j + P_{ml\hat{k}}^{+-+}\xi_{mi}^{+}\xi_{lj}^{-}n_j + P_{ml\hat{k}}^{-++}\xi_{mi}^{-}\xi_{lj}^{+}n_j + P_{ml\hat{k}}^{--+}\xi_{mi}^{-}\xi_{lj}^{-}n_j \\
&+ \frac{1}{Re}T_{\hat{k}}^{+}\left(u_{Bi,j}^{+} + u_{Bi,j}^{-}\right)n_j + \frac{1}{Re}\left(S_{\hat{k}l}^{++}\xi_{li,j}^{+} + S_{\hat{k}l}^{+-}\xi_{li,j}^{-}\right)n_j = 0
\end{aligned}
\tag{8a}
$$

$$
\begin{aligned}
&P_{ml\hat{k}}^{---}\xi_{mi}^{-}\xi_{lj}^{-}n_j + P_{ml\hat{k}}^{-+-}\xi_{mi}^{-}\xi_{lj}^{+}n_j + P_{ml\hat{k}}^{+--}\xi_{mi}^{+}\xi_{lj}^{-}n_j + P_{ml\hat{k}}^{++-}\xi_{mi}^{+}\xi_{lj}^{+}n_j \\
&+ \frac{1}{Re}T_{\hat{k}}^{-}\left(u_{Bi,j}^{+} + u_{Bi,j}^{-}\right)n_j + \frac{1}{Re}\left(S_{\hat{k}l}^{--}\xi_{li,j}^{-} + S_{\hat{k}l}^{-+}\xi_{li,j}^{+}\right)n_j = 0
\end{aligned}
\tag{8b}
$$

For Robin boundary conditions:

$$
\begin{aligned}
&S_{m\hat{k}}^{++}\xi_{li}^{+}\left(u_{Bj}^{+} + u_{Bj}^{-}\right)n_j + S_{m\hat{k}}^{-+}\xi_{li}^{-}\left(u_{Bj}^{+} + u_{Bj}^{-}\right)n_j \\
&+ \frac{1}{Re}T_{\hat{k}}^{+}\left(u_{Bi,j}^{+} + u_{Bi,j}^{-}\right)n_j + \frac{1}{Re}\left(S_{\hat{k}l}^{++}\xi_{li,j}^{+} + S_{\hat{k}l}^{+-}\xi_{li,j}^{-}\right)n_j = 0
\end{aligned}
\tag{9a}
$$

$$
\begin{aligned}
&S_{m\hat{k}}^{--}\xi_{li}^{-}\left(u_{Bj}^{+} + u_{Bj}^{-}\right)n_j + S_{m\hat{k}}^{+-}\xi_{li}^{+}\left(u_{Bj}^{+} + u_{Bj}^{-}\right)n_j \\
&+ \frac{1}{Re}T_{\hat{k}}^{-}\left(u_{Bi,j}^{+} + u_{Bi,j}^{-}\right)n_j + \frac{1}{Re}\left(S_{\hat{k}l}^{--}\xi_{li,j}^{-} + S_{\hat{k}l}^{-+}\xi_{li,j}^{+}\right)n_j = 0
\end{aligned}
\tag{9b}
$$

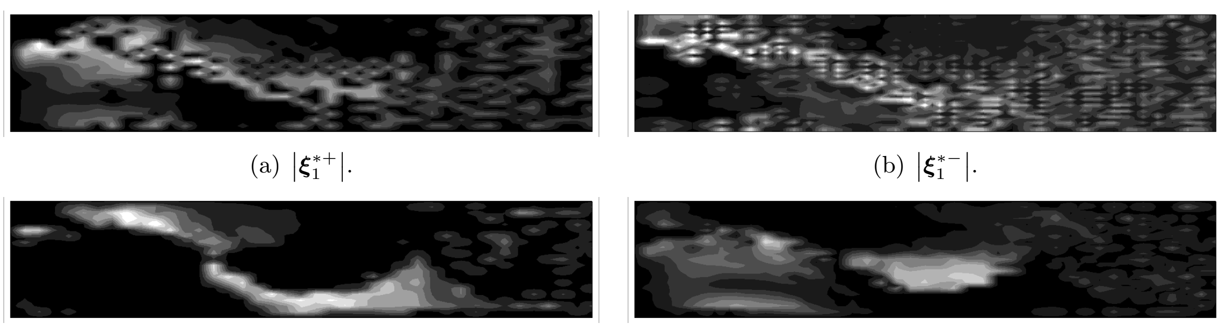
Where,

$$\begin{cases} B_{\hat{k}lm}^{\pm\pm\pm} = \int_\Omega \xi_{\hat{k}i}^{\pm}\xi_{lj}^{\pm}\xi_{mi,j}^{\pm}\ \mathrm{d}\Omega \\ C_{\hat{k}l}^{\pm\pm} = \int_\Omega \xi_{\hat{k}i,j}^{\pm}\xi_{li,j}^{\pm}\ \mathrm{d}\Omega \\ E_{\hat{k}l}^{\pm\pm} = \int_\Omega \xi_{\hat{k}i}^{\pm}\xi_{li}^{\pm}\ \mathrm{d}\Omega \\ F_{\hat{k}l}^{\pm\pm} = \int_\Omega \varepsilon_{ijk}\xi_{\hat{k}i}^{\pm}\xi_{lk,j}^{\pm}\ \mathrm{d}\Omega \\ P_{\hat{k}lm}^{\pm\pm\pm} = \int_0^T \tilde{\lambda}_{\hat{k}}^{\pm}a_l^{\pm}a_m^{\pm}\ \mathrm{d}t \\ S_{\hat{k}l}^{\pm\pm} = \int_0^T \tilde{\lambda}_{\hat{k}}^{\pm}a_l^{\pm}\ \mathrm{d}t \\ T_{\hat{k}}^{\pm} = \int_0^T \tilde{\lambda}_{\hat{k}}^{\pm}\ \mathrm{d}t \end{cases} \tag{10}$$

With the multi-scale global optimization method [5], we can solve the above ODEs and obtain the optimal generalized helical-wave bases $\xi_{\hat{k}i}^{*+}$ and $\xi_{\hat{k}i}^{*-}$ directly from Navier-Stokes equations.

3 Optimal generalized helical-wave bases and their properties

For the numerical simulation example, we choose the backward facing step flow with Reynolds number $Re = 800$. Then we construct the coupling dynamical systems and obtain six pairs of optimal generalized helical-wave bases $\boldsymbol{\xi}_{\hat{k}}^{*+}$ and $\boldsymbol{\xi}_{\hat{k}}^{*-}$, ($\hat{k} = 1, 2, \cdots, 6$), and the contour lines of the magnitudes of $\boldsymbol{\xi}_1^{*\pm}$ and $\boldsymbol{\xi}_2^{*\pm}$ in the streamwise middle section of the channel is shown in figure 1.



(a) $|\boldsymbol{\xi}_1^{*+}|$. (b) $|\boldsymbol{\xi}_1^{*-}|$.

(c) $|\boldsymbol{\xi}_2^{*+}|$. (d) $|\boldsymbol{\xi}_2^{*-}|$.

Fig. 1: The contour lines of the magnitudes of $\boldsymbol{\xi}_1^{*\pm}$ and $\boldsymbol{\xi}_2^{*\pm}$ in the streamwise middle section.

From these pictures, we can find the properties of optimal generalized helical-wave bases as follows.

(1) Optimal generalized helical-wave bases reflect the properties of smaller scale structures, especially compared with POD bases.

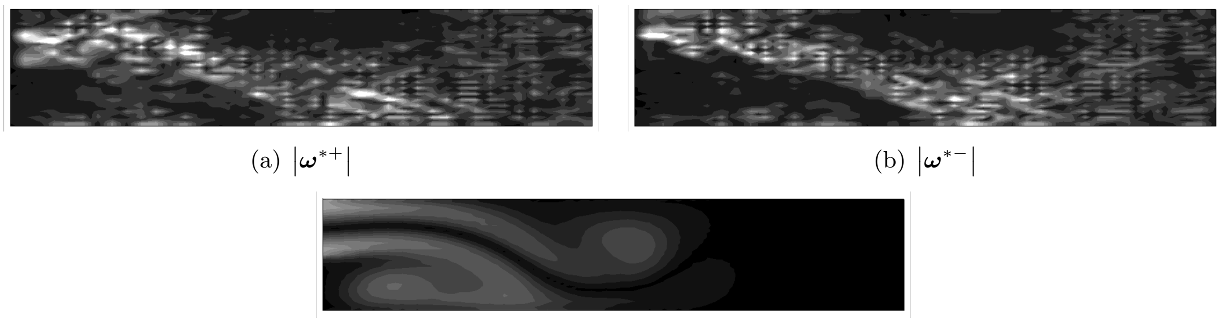
(2) They show the parity property and the coupling interaction between the positive one $\boldsymbol{\xi}_{\hat{k}}^{*+}$ and the negative one $\boldsymbol{\xi}_{\hat{k}}^{*-}$.

(3) They are more complex and broken in the spatial distribution, which contain the properties of Beltrami flow's chaotic particles.

(4) They do not distribute continuously in physical space, which is the most important difference with POD bases, and they are concentrated.

4 Optimal generalized helical-wave vortexes – the fundamental elements of turbulence

Then we use the six pairs of optimal generalized helical-wave bases $\boldsymbol{\xi}_{\hat{k}}^{*+}$ and $\boldsymbol{\xi}_{\hat{k}}^{*-}$ to fit the positive component and negative component of the vorticity fields and obtain $\boldsymbol{\omega}^{*+}$ and $\boldsymbol{\omega}^{*-}$ respectively satisfying left-handed and righted-handed parity symmetry properties, which we called the optimal helical-wave vortexes. The contour lines of the magnitudes of $\boldsymbol{\omega}^{*+}$, $\boldsymbol{\omega}^{*-}$ and the original vorticity field $\boldsymbol{\omega}$ at the same moment in the streamwise middle section is shown in figure 2.



(a) $|\boldsymbol{\omega}^{*+}|$ (b) $|\boldsymbol{\omega}^{*-}|$ (c) $|\boldsymbol{\omega}|$

Fig. 2: The contour lines of the magnitudes of $\boldsymbol{\omega}^{*+}$, $\boldsymbol{\omega}^{*-}$ and $\boldsymbol{\omega}$ at the same moment in the streamwise middle section.

From figure 2, we can find that:

(1) $\boldsymbol{\omega}^{*+}$ and $\boldsymbol{\omega}^{*-}$ show small scale vortex structures of the vorticity field, and their interaction and superposition result in large vortex structures in $\boldsymbol{\omega}$.

(2) $\boldsymbol{\omega}^{*+}$ and $\boldsymbol{\omega}^{*-}$ are concentrated in physical space, and in some locations the vorticity magnitudes $|\boldsymbol{\omega}^{*+}|$ and $|\boldsymbol{\omega}^{*-}|$ are much bigger than $|\boldsymbol{\omega}|$, which reflects the high polarization effect from large scale vortexes to small scale ones.

Then when considering the evolution over time, we find that $\boldsymbol{\omega}^{*+}$ and $\boldsymbol{\omega}^{*-}$ change much more slowly than $\boldsymbol{\omega}$, so i.e. the interaction and superposition of $\boldsymbol{\omega}^{*+}$ and $\boldsymbol{\omega}^{*-}$ which change slowly result in the fast changing of $\boldsymbol{\omega}$. In this way, it is proved that optimal generalized helical-wave bases meet the three properties mentioned in section 1, so they can be used as the fundamental element of turbulence.

5 Conclusion

In this paper, we present the theory of constructing optimal generalized helical-wave coupling dynamical systems and obtain the optimal generalized helical-wave bases, which have many good properties. On the one hand, they retain the good properties of classical helical-wave bases; on the other hand, they are global optimal for dynamical systems of Navier-Stokes equations, and suitable for complex physical and geometric boundary conditions. Then we find that the optimal generalized helical-wave vortexes fitted by the optimal generalized helical-wave bases can be used as the fundamental elements of turbulence, and have important significance for studying physical properties of complex flows and turbulent vortex structures in a deeper level.

Acknowledgements. This work was supported by the National Natural Science Foundation of China grant No.11372068, the National Basic Research Program of China grant No.2014CB744104 and the National Natural Science Foundation of China grant No.11572350.

References

[1] Wu, C.J. & Zhao, H.L. (2000) Generalized HWD-POD method and coupling low-dimensional dynamical systems on turbulence. *Discrete Contin. Dyn. Syst.* 2000 **7 (Added Volumn)**, 371–379.

[2] Moses, H.E. (1971) Eigenfunctions of the curl operator, ratationally invarient Helmholtz theorem, and applications to electromagnetic theory and fluid mechanics. *SIAM J. Appl. Math.* 1971 **21**, 114–144.

[3] Lesieur, M. (1990) *The Physics of Fluid Turbulence.* Oxford University Press, Oxford.

[4] Fu, D.X. & Ma, Y.W. & Li, X.L. & Wang, Q. (2010) *Direct Numerical Simulations of Compressible Turbulence* (in Chinese). Science Press, Beijing.

[5] Peng, N. F. & Guan, H. & Wu, C. J. Research on the optimal dynamical systems of three-dimensional Navier-Stokes equations based on weighted residual. submitted to *Sci. China Ser. G-Phys. Mech. Astron.*