Topology of flows and finite-time singularity: On the kernel of helicity and spirality

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Summary

The aim of this note is to draw a connection between topology and geometry of fluid flows and finite time singularity in vanishing helicity fluids. Our inquiry leads to characterizing the vortex flows according to their knottedness and integrability of the velocity field. Since the helicity is zero, we will use polynomial invariants on vortex lines to distinguish different systems in our examples. The hope to use more mathematical results inspired us to redefine helicity and spirality in more general forms as operators acting on differential forms on odd dimensional smooth manifolds. The fluid dynamical assumptions and previous results will be transformed in the terms of differential forms. In the next step, we will use a new cohomological structure to redefine the operators on topological manifolds.

1 Introduction

The zero helicity for ideal flows neither characterizes the vortex structure nor bounds the enstrophy (i.e., energy of the vorticity field). This situation recently investigated by more powerful tools, from constructing higher-helicity via Massey product and Milnor μ invariant toward using the polynomial knot invariants [1]–[3]. At first we focus on this kind of flows and try to find the relation between possibility of happening the finite-time singularity and existence a gauge transformation on the impulse function $\gamma = u - \nabla\phi$, where u is the velocity of fluid and ϕ is the Bernoulli's function. Sprirality, that is a Lagrangian invariant is defined in the Russo notion as follows

$$Sp = \frac{\gamma \cdot \omega}{\rho}, \tag{1}$$

where ω is vorticity and ρ is fluid density [4]. The above quantity is only depend on the initial condition and therefore remains constant on the fluid trajectory for a fluid particle. The Integral of $\gamma \cdot \omega$ on a domain of fluid D (with $\omega \cdot n = 0$ on ∂D) is the helicity. Obviously any gauge transformation as $\gamma \to \gamma - \nabla\varphi_0$ such that φ_0 be a time independent function of initial position, changes the spirality but holds the vorticity and helicity of the fluid. The main question of this note is finding the topological and geometrical conditions of the initial fluid flows that guarantee the existence of a gauge transformation to bring the spirality to zero [5]. It seems, having this kind of gauge increase the possibility of occuring the singularity for vorticity field on the finite number of points on the trajectory of the fluid particles, in finite intervals of time. Although the complete answer to this question is not available now, we have two separate necessary and sufficient conditions for the existence of this gauge. The necessary condition corresponds to the knottedness of the vortex lines such that the total flux passing through any embedded Seifert surface of a non singular vortex lines should be zero. We will show this behavior with a few examples of vortex tubes tabulated by Alexander and Jones polynomials. The sufficient condition tell us if the velocity field (harmonic part of velocity in high genus boundary surface) satisfies the Frobenius integrability condition (i.e., have a collection of locally normal surfaces) the gauge is principally exist.

The hope to use some mathematical achievements in the literature, inspired us to generalize the helicity and spirality to operators acting on a quotient class of differential forms on three dimensional smooth manifolds and reconstruct the necessary and the sufficient conditions in the new format [6]. In this condition, if $j_M : \partial M \hookrightarrow M$ be the inclusion map, $j_M^*(d\alpha) = 0$ (or $\iota_{\frac{\partial}{\partial\xi}}(*d\alpha) = 0$ for any $\frac{\partial}{\partial \xi}$ normal to the boundary ∂M) describe the tangent to the boundary property of the vector field related to the form $d\alpha$. Now if we defined the subset $\Omega_T^p(M)$ of $\Omega^p(M)$ as the exact forms $d\omega$ with tangent property, the spirality operator Sp on $2p+1$ manifold M defined as the map from $\Omega_T^p(M)/Im(d)$ to $\Omega^{2p+1}(M)/\Omega_T^{2p+1}(M)$, sending the class of p-forms $[\alpha]_{\Omega_T^p(M)/Im(d)}$ to the class of top degree forms $[\alpha \wedge d\alpha]_{\Omega^{2p+1}(M)/\Omega_T^{2p+1}(M)}$ and helicity defined as $\int_M Sp([\alpha])$. Again the transformation $\alpha \to \alpha + d\beta$ for a given p-form β holds the helicity.

A smooth map $f : N \to M$ pullbacks the above structures from M to N. Therefore, for $g := f|_{\partial N} : \partial N \to \partial M$ we have the commutative relation $f \circ j_N = j_M \circ g$. Here the problem of existing the gauge in the special case of vanishing helicity giving rise to the difference between the kernels of helicity and spirality.

In the next step, the previous objects translated in the language of a special class of singular cochains on topological manifolds and the standard cup product $\smile$ modified to a new form $\overline{\smile}$ to be compatible with the wedge product $\wedge$ of forms in smooth condition ([6], [7]). The spirality becomes as follows

$$Sp(\alpha) = \alpha \overline{\smile} \delta\alpha, \tag{2}$$

where δ is the boundary operator. By applying the above helicity on the fundamental class, we reach to the helicity of singular cochains. The obtained cohomology in the above argument can make a platform for new topological version of more invariants e.g., triple-product, Chern-Simons action and also the Calabi invariant that is the even dimensional sibling of helicity. The compatibility of each singular quantity with the original smooth version, will be checked by using the isomorphism between de Rham cohomology and singular cohomology [8].

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