Field line helicity and magnetic reconnection

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Summary

It is often useful to know not only the global helicity of a magnetic field, but also how the total is composed and how that changes over time. The tool that best achieves this is *field line helicity*, which assigns a helicity value to every field line and can therefore distinguish between different magnetic fields with the same total helicity. Here, I will discuss the properties of field line helicity, demonstrate its practical application to magnetic fields, and (the main result) show how an evolution equation can be derived for it. Finally, the evolution of field line helicity is characterised for two important cases: ideal evolution and localised reconnection in a complex magnetic field. The latter has interesting implications for Taylor relaxation.

1 Introduction

Magnetic helicity is a valuable tool that quantifies the overall linking of magnetic field lines and can be extended to open domains either by using relative helicity or by suitable restriction of the gauge. This global measure of the topology is complemented by a local measure that we refer to as the *field line helicity* [1, 2].

2 Definition and example

For field lines that close, or that enter and exit the domain, the field line helicity at $\mathbf{x}$ is defined as

$$\mathcal{A}(\mathbf{x}) = \int_{F(\mathbf{x})} \mathbf{A} \cdot \mathrm{d}\mathbf{l} \qquad (1)$$

where $\mathbf{A}$ is the vector potential, $\mathrm{d}\mathbf{l}$ is the line element parallel to $\mathbf{B} = \nabla \times \mathbf{A}$, and the integral is over the field line through the point of interest. The total helicity of any flux tube can be obtained from $\mathcal{A}$ by evaluating the flux integral over any cross-section, C, of the tube, since

$$H = \int_V \mathbf{A} \cdot \mathbf{B} \, \mathrm{d}^3x = \int_C \mathcal{A}\mathbf{B} \cdot \mathbf{n} \, \mathrm{d}^2x, \qquad (2)$$

where $\mathbf{n}$ is the unit normal to the cross-section; this property justifies the name *field line helicity.*

$\mathcal{A}$ is valuable because it is an ideal invariant (unlike $\mathbf{A} \cdot \mathbf{B}$) and it retains more topological information than the total helicity. Figure 1 demonstrates the second point – the braided magnetic field shown has zero total helicity but $\mathcal{A}$ captures its considerable topological complexity.

3 Evolution

For any Ohm's law, a *generalised field line velocity*, $\mathbf{w}$, exists such that

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{w} \times \mathbf{B}) \qquad (3)$$

i.e. $\mathbf{B}$ evolves as though frozen in to the flow of $\mathbf{w}$ (here, we exclude null points, spines and fan surfaces). In the absence of parallel electric field, $\mathbf{w}$ reduces to the plasma velocity; when

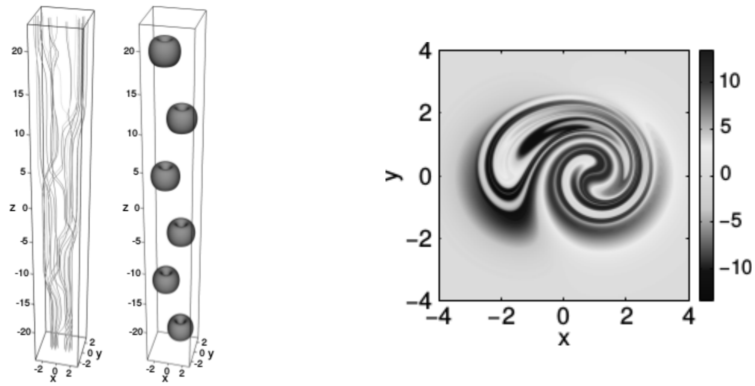

Fig. 1: (Left) Visualisation of the E^3 magnetic braid [3]: the two boxes show (leftmost) a selection of the field lines and (adjacent) the six flux rings that are superimposed on a uniform vertical field to create the braid. (Right) Field line helicity on the base of the domain.

reconnection occurs, $\mathbf{w}$ also captures the apparent slipping of field lines with respect to the plasma. One can then show [4] that for a suitable restriction of the gauge,

$$\frac{\partial \mathcal{A}}{\partial t} = -\mathbf{w} \cdot \nabla \mathcal{A} + \mathbf{w} \cdot \mathbf{A}|_{\text{exit}} - \mathbf{w} \cdot \mathbf{A}|_{\text{entry}} - \Delta\psi, \tag{4}$$

where $\Delta\psi$ is the voltage drop along the field line and the $\mathbf{w} \cdot \mathbf{A}|_{\text{entry}}$ and $\mathbf{w} \cdot \mathbf{A}|_{\text{exit}}$ terms are evaluated where the field line meets the boundary (these terms are dropped for closed field lines).

For ideal evolutions, $\Delta\psi = 0$. If the plasma is static at the boundaries then the $\mathbf{w} \cdot \mathbf{A}$ terms also vanish and Eq. (4) reduces to an advection equation. Hence, $\mathcal{A}$ is an ideal invariant. If motions are permitted on the boundaries, then these displacements change the winding of magnetic field lines and this topology change is captured in the evolution of $\mathcal{A}$ due to the "work-like" $\mathbf{w} \cdot \mathbf{A}$ terms.

In general, when reconnection occurs the full form of Eq. (4) applies. However, a powerful simplification is obtained in fields that have significant topological entropy, e.g. as is typical for turbulent reconnection. Then, the terms on the RHS have a useful ordering: the advection term dominates; followed by the $\mathbf{w} \cdot \mathbf{A}$ terms, which have a well-ordered structure of oppositely signed pairs; and the voltage drop term makes the least contribution. Thus, to leading order reconnection rearranges field line helicity, with the next order effect being small pairwise exchanges.

An interesting implication is that while turbulent reconnection approximately conserves total helicity, it cannot arbitrarily change how the helicity is composed, contrary to the assumption underlying standard Taylor relaxation. We therefore suggest that more accurate estimates of the relaxation end-state and the energy released may be obtained by using this more rigorous constraint.

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