

# The elastic trefoil is the twice covered circle

H. Gerlach<sup>1</sup>, Ph. Reiter<sup>2</sup>, H. von der Mosel<sup>3</sup>

<sup>1</sup> *LCVMM, EPF Lausanne, Switzerland*

<sup>2</sup> *Faculty of Mathematics, University of Duisburg–Essen, Germany*

<sup>3</sup> *Institute for Mathematics, RWTH Aachen University, Germany*

## Summary

In order to investigate the elastic behavior of knotted loops of springy wire, we minimize the classic bending energy regularized by ropelength, i.e., the quotient of length over thickness, in order to penalize self-intersection. Our main objective is to characterize the limit configurations of energy minimizers as the regularization parameter tends to zero, which will be referred to as *elastic knots*.

For every odd  $b > 1$  and the respective class of  $(2, b)$ -torus knots (containing the trefoil) we obtain a complete picture showing that the respective elastic  $(2, b)$ -torus knot is the twice covered circle.



Fig. 1: Springy knots: figure-eight knot, mathematician’s loop, and Chinese button knot. Wire models manufactured by WHY KNOTS, Aptos; photographs by B. Bollwerk, Aachen.

Knotted loops made of elastic wire spring into some (not necessarily unique) stable configurations when released, such as the beautiful toy models shown in Figure 1. The aim of this project is to characterize the corresponding shapes.

Neglecting any additional torsional effects and also excluding external forces and friction that might be present these toy models, we follow Bernoulli’s approach to consider the *bending energy*

$$E_{\text{bend}}(\gamma) := \int_{\gamma} \kappa^2 ds \quad (1)$$

as the only intrinsic elastic energy. Here the wire is represented by a sufficiently smooth *unit loop*, i.e., a closed curve  $\gamma : \mathbf{R}/\mathbf{Z} \rightarrow \mathbf{R}^3$  of unit length, parametrized by arclength. Its local curvature is denoted by  $\kappa = |\gamma''|$ .

To respect a given knot class when minimizing the bending energy we have to preclude self-crossings. As proposed in [1], we could add any self-repulsive *knot energy* for that matter, imposing infinite energy barriers between different knot classes. But a solid (albeit thin) wire motivates a steric constraint in form of a fixed (small) thickness of all curves in competition. This and the geometric rigidity it imposes on the curves lead us to adding a small amount of *ropelength*  $\mathcal{R}$  to form the *total energy*

$$E_{\vartheta} := E_{\text{bend}} + \vartheta \mathcal{R}, \quad \vartheta > 0, \quad (2)$$

to be minimized within a prescribed (tame) knot class  $\mathcal{K}$ , that is, on the class  $\mathcal{C}(\mathcal{K})$  of all unit loops representing  $\mathcal{K}$ . The thickness  $\Delta[\cdot]$  may be expressed as

$$\Delta[\gamma] := \inf_{\substack{u, v, w \in \mathbf{R}/\mathbf{Z} \\ u \neq v \neq w \neq u}} R(\gamma(u), \gamma(v), \gamma(w)), \quad (3)$$

where  $R(x, y, z)$  denotes the unique (possibly degenerate) circle passing through  $x, y, z \in \mathbf{R}^3$ .



Fig. 2: Minimizers of  $E_\vartheta$ ,  $\vartheta = 0.1$  (left) and  $\vartheta = 0.001$  (right), for the trefoil knot class. The figures have been produced by simulated annealing.

It can be shown that in every given (tame) knot class  $\mathcal{K}$  and for every  $\vartheta > 0$  there is indeed a unit loop  $\gamma_\vartheta \in \mathcal{C}(\mathcal{K})$  minimizing the total energy  $E_\vartheta$  within  $\mathcal{K}$ , see [3, Thm. 2.1].

To understand the behaviour of very thin springy knots we investigate the limit  $\vartheta \searrow 0$ , see [1]. To this end, we consider arbitrary sequences  $(\gamma_\vartheta)_{\vartheta>0}$  of minimizers in a fixed knot class  $\mathcal{K}$  and look at their possible limit curves  $\gamma_0$  as  $\vartheta \searrow 0$ , calling any such limit curve an *elastic knot for  $\mathcal{K}$* . For any tame knot class, there is at least one elastic knot. Unless  $\mathcal{K}$  is the unknot, none of these elastic knots is embedded.

In [3] we rigorously prove that, for any non-trivial knot class attaining the smallest possible lower bound  $(4\pi)^2$  on the bending energy, the *only* possible shape of an elastic knot is that of the twice-covered circle. Moreover, we show that the former condition holds *exactly* for the class of  $(2, b)$ -torus knots for any odd integer  $b$  with  $|b| \geq 3$ .

This result confirms our numerical and mechanical experiments (see Figure 2 and Figure 3 on the left), as well as the heuristics and simulations of Gallotti and Pierre-Louis [2].

Twisting the wire in the experiments before closing it at the hinge (without releasing the twist) leads to completely different stable configurations; see Figure 3 on the right. In that case a more general Lagrangian taking into account also these effects would need to be considered, and the question of classifying *twisted* elastic knots is wide open.



Fig. 3: Mechanical experiments. Left: The springy trefoil knot is close to the twice-covered circle. Right: Adding a  $2\pi$ -twist leads to a stable flat trefoil configuration close to a planar figure-eight. Wire models by courtesy of J. H. Maddocks, Lausanne.

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## References

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