Singularities in viscous flows

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Summary

Available steady solutions of Navier–Stokes equations with singularities are reviewed, and new examples of such solutions are given. Examples of unsteady solutions are also given, which demonstrate the change in the flow region topology or the structure of streamlines with time. The problem of deformation of a strip bounded by a solid wall and a parallel free boundary is studied. Conditions of solution failure within a finite time are found.

This work consists of two parts. The first one is a review of steady solutions of Navier–Stokes equations for an incompressible fluid with singularities [1]. They include the classical solutions obtained by Jeffery, Hamel, Slezkin, Landau, Goldshtik, and also the solutions obtained by the authors of the present text in [2] and [3], which describe plane and axisymmetric flows with a free surface. We also consider the asymptotic character of singularities of the source, sink, or point vortex type. In particular, we consider the problem of plane motion of a viscous fluid in an arbitrary bounded domain with a solid boundary. This motion is induced by rotation of a solid disk of small radius α with an angular velocity ω inside the domain. It is proved that the solution of the point vortex problem can be approximated in a suitable metrics by the solution of the rotating disk problem as $\alpha \to 0$ and $\omega \to \infty$, so that $\alpha^2\omega \to |\Gamma|$ [4].

The problem of a rotating ring is a seldom example of the problem where the time evolution of the flow topology can be traced [5], [6]. In this solution, the free boundaries are circumferences, and the solution of two-dimensional Navier–Stokes equations is independent of the polar angle. The problem has two integrals of motion: S (ring area) and M (moment of momentum). Let us denote $\beta = \left(\rho\sigma S^{5/2}\right)^{-1} L^2$, where ρ is the fluid density and σ is the surface tension coefficient. Let us assume that $\beta < \beta^* \approx 0.831$. Then the rotating ring transforms to a circle within a finite time; moreover, this process is irreversible. [5].

Let us give an example of the flow where the structure of streamlines changes instantaneously [7]. Let the fluid fill the sector $0 < \varphi < \alpha$ with solid boundaries, and let the stream function of the plane flow at the time $t = 0$ be constant on each ray $\varphi = const$. The resultant motion is self-similar and serves as a two-dimensional extension of the classical Jeffery–Hamel solution. If the flow rate of the fluid through the origin is equal to zero and the angle α is sufficiently small, then the information about the initial velocity distribution is entrained to infinity as $t \to \infty$, and a sequence of vortices originates from the origin. These vortices become decelerated with time, and their shape approaches the famous Moffatt vortices [8].

To conclude, we consider a plane problem where one velocity component is a linear function of the x coordinate and the other velocity component and the pressure are independent of x. The proposed solution is interpreted as unsteady motion in a strip bounded by the solid wall $y = 0$ and the free boundary $y = a(t)$. The problem has a self-similar solution where $y/\sqrt{t} = const$ [9]. The numerical analysis of the problem shows that there may exist two stable regimes in addition to the unstable self-similar regime, depending on the initial data [10]. In one of them, $a \to const$ as $t \to \infty$, and the motion stabilizes to the quiescent state. In the other regime, $a \to \infty$ as $t \to t^* < \infty$. The asymptotic behavior of the collapsing solution is described by the relation $a(t - t^*) \to const$ typical for solutions of Euler equations with a straight free boundary [11], [12].

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