

Helicity-Enhanced Extreme Vortex States and the Hydrodynamic Blow-Up Problem

B. Protas

Department of Mathematics & Statistics, McMaster University, Hamilton, ON, L8S 4K1, Canada

Summary

We present a variational framework for analyzing the effect of helicity on the structure and properties of extreme vortex states in viscous incompressible flows. These states arise as the flows exhibiting the largest growth of enstrophy allowed by the system dynamics under some imposed constraints and are therefore intimately related to the question of the possible singularity formation in the 3D Navier-Stokes and Euler systems, known as the “blow-up problem”. A key element of the approach is a multiobjective formulation of the underlying optimization problem. In the presentation we will review details of this formulation and survey some of the properties of the helicity-enhanced extreme vortex states.

1 Introduction

The *extreme vortex states* are defined as the flows saturating certain fundamental mathematical estimates, such as the bounds on the maximum enstrophy growth in 3D [1], in the presence of suitable constraints on the flow field. Similar problems are also relevant in the context of the 1D Burgers and 2D Navier-Stokes systems. While these systems are known *not* to lead to singularity formation in finite time, the question of the accuracy of their worst-case estimates is still important, as these bounds are obtained using analogous methods as in the 3D case. By framing these questions in terms of variational PDE optimization problems, which are solved computationally using discrete gradient flows, we are able to provide new insights concerning the “sharpness” of these estimates, understood as the existence of actual vector fields which realize these bounds [2, 3, 4, 5]. In offering a systematic approach to finding flow solutions which may saturate known estimates, this paradigm thus provides a bridge between rigorous mathematical analysis and scientific computation.

2 Estimates of Enstrophy Growth

As regards flows in 3D unbounded or periodic domains $\Omega \subseteq \mathbb{R}^3$, the Navier-Stokes system is known to admit classical solutions $\mathbf{u}(t, \mathbf{x})$ which are *globally* well-posed as long as the enstrophy $\mathcal{E}(\mathbf{u}(t)) := \int_{\Omega} |\nabla \times \mathbf{u}(t, \mathbf{x})|^2 d\Omega$ remains bounded. At the same time, the best upper bound on the instantaneous rate of growth of enstrophy is

$$\frac{d\mathcal{E}}{dt} \leq C \mathcal{E}^3 \quad (1)$$

with some $C > 0$. By solving variational problems of the type

$$\tilde{\mathbf{u}}_{\mathcal{E}_0} = \arg \max_{\mathbf{u} \in H^1(\Omega)} \mathcal{R}(\mathbf{u}) \quad \text{subject to: } \mathcal{E}(\mathbf{u}) = \mathcal{E}_0, \quad (2)$$

where $\mathcal{R}(\mathbf{u}) := d\mathcal{E}(\mathbf{u})/dt = -\nu \int_{\Omega} |\Delta \mathbf{u}|^2 dx + \int_{\Omega} \mathbf{u} \cdot \nabla \mathbf{u} \cdot \Delta \mathbf{u} dx$, for a range of different values of \mathcal{E}_0 . Lu & Doering [1] found a family of divergence-free vector fields $\tilde{\mathbf{u}}_{\mathcal{E}_0}$ saturating the upper bound in (1), thereby demonstrating that this instantaneous estimate is sharp. However, as shown in [5], these extreme vortex states are characterized by a very simple structure involving two colliding, nearly axisymmetric vortex rings (Fig. 1a). In fact, these optimal vector fields have the structure of generalized Beltrami flows with vanishing helicity $\mathcal{H}(\tilde{\mathbf{u}}_{\mathcal{E}_0}) = 0$, where $\mathcal{H}(\mathbf{u}) := \int_{\Omega} \mathbf{u} \cdot (\nabla \times \mathbf{u}) d\Omega$. As a result, the initially large rate of growth of enstrophy saturating (1) is not sustained yielding only a modest increase of enstrophy in finite time. Maintaining a rapid rate of growth $d\mathcal{E}(\mathbf{u})/dt$, sufficient for the enstrophy $\mathcal{E}(t)$ to become unbounded in finite time, is a necessary precursor of blow-up in the 3D Navier-Stokes system [1].

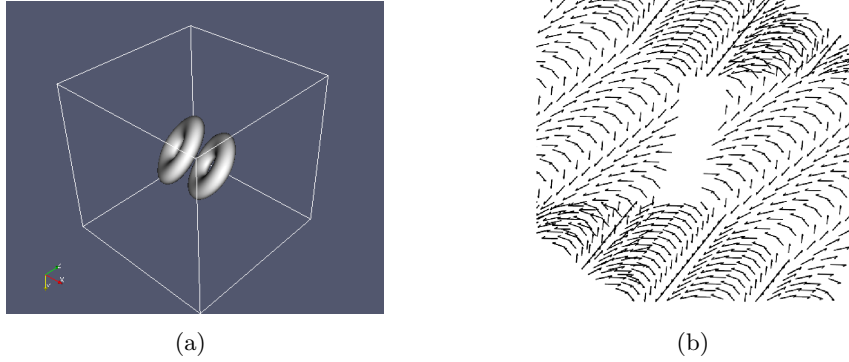


Fig. 1: (a) Extreme vortex state $\tilde{\mathbf{u}}_{\mathcal{E}_0}$ with $\mathcal{E}_0 = 20$ [5] (corresponding to solutions of (2), or (3) with $\alpha = 1$). (b) Eigenfunction of the *curl* operator maximizing helicity $\mathcal{H}(\mathbf{u})$ for given enstrophy \mathcal{E}_0 (corresponding to solutions of (3) with $\alpha = 0$).

3 Extreme Vortex States with Finite Helicity

To produce a larger sustained growth of enstrophy, it is therefore necessary to construct optimal vortex states with nontrivial structure which can be achieved by ensuring that they possess nonzero helicity $\mathcal{H}(\mathbf{u})$. It is known that the growth of enstrophy at a rate somewhat slower than given in (1), namely $d\mathcal{E}(t)/dt \sim CE^\beta$ for some $\beta > 2$, is still sufficient to produce blow-up in finite time. Thus, to push the limit of the maximum enstrophy growth, it is necessary to construct extreme vortex states $\tilde{\mathbf{u}}_{\mathcal{E}_0, \mathcal{H}}^\alpha$ characterized by a suboptimal instantaneous rate of growth of enstrophy (i.e., with $2 < \beta < 3$), but at the same time having a nontrivial topological structure reflected in nonzero helicity $\mathcal{H}(\tilde{\mathbf{u}}_{\mathcal{E}_0, \mathcal{H}}^\alpha) \neq 0$. The latter property will ensure that a rapid rate of growth of enstrophy can be sustained over longer time windows. Such states can be constructed based on a variational formulation relying on multiobjective optimization which generalizes (2). Given parameter $\alpha \in [0, 1]$, the new objective function blending helicity and the instantaneous rate of enstrophy growth can be defined as $\mathcal{Q}_\alpha(\mathbf{u}) := \alpha\mathcal{R}(\mathbf{u}) + (1 - \alpha)\mathcal{H}(\mathbf{u})$, so that the corresponding optimization problem becomes

$$\tilde{\mathbf{u}}_{\mathcal{E}_0, \mathcal{H}}^\alpha = \arg \max_{\mathbf{u} \in H^1(\Omega)} \mathcal{Q}_\alpha(\mathbf{u}) \quad \text{subject to: } \mathcal{E}(\mathbf{u}) = \mathcal{E}_0. \quad (3)$$

Thus obtained extreme vortex states $\tilde{\mathbf{u}}_{\mathcal{E}_0, \mathcal{H}}^\alpha$ will constitute a one-parameter family of solutions known as the ‘‘Pareto front’’. They connect two limiting solutions, namely, the original extreme vortex states $\tilde{\mathbf{u}}_{\mathcal{E}_0}$ maximizing $d\mathcal{E}(t)/dt$, cf. (1), are recovered when $\alpha = 1$, whereas for $\alpha = 0$ we obtain fields which for given enstrophy $\mathcal{E}(\mathbf{u})$ possess the largest possible helicity $\mathcal{H}(\mathbf{u})$. It can be demonstrated that these fields, illustrated in Fig. 1b, are given in terms of the eigenfunctions of the *curl* operator associated with the smallest nonzero eigenvalue.

References

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