A definative definition of the open field magnetic helicity

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Summary

The magnetic helicity is invariant under ideal flows which vanish at the domain boundary. However, the quantity is not gauge invariant where there are open field lines. Berger and Field first proposed a meaningful definition of this quantity [1] which was gauge invariant but depended on comparison to a fictitious reference field. Since then there has been a significant number of authors proposed alternative definitions of the open helicity, but none have been shown to be unambiguously preferable. We present published work [2] showing all these definitions can be interpreted in terms of the average pair-wise net winding of the fields curves (an open analogue of the linking number) and that all but one one possible definition (choice of gauge) have some non-physical contribution. New results presented will show this definition can be used to uniquely classify braided magnetic fields and hence accurately characterise magnetic re-connection.

Magnetic helicity $H(\mathbf{B}) = \int_V \mathbf{A}\cdot\mathbf{B}\,\mathrm{d}^3x$, with $\mathbf{B} = \nabla\times\mathbf{A}$, has long been recognized as an important dynamical invariant in ideal magnetohydrodynamics. The integral $H(\mathbf{B})$is independent of the particular gauge chosen for $\mathbf{A}$, provided that V is simply connected and magnetically closed ($B_n = 0$ on the boundary ∂V) [3].

Physically, $H(\mathbf{B})$ may be interpreted as the flux-weighted average, over all pairs of magnetic field lines $\mathrm{d}\mathbf{x}/\mathrm{d}s = \mathbf{B}(\mathbf{x})$, $\mathrm{d}\mathbf{y}/\mathrm{d}s = \mathbf{B}(\mathbf{y})$, of the Gauss linking integral

$$L(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi}\oint_{\mathbf{x}(s)}\oint_{\mathbf{y}(s)}\frac{d\mathbf{x}}{ds}\cdot\frac{d\mathbf{y}}{ds}\times\frac{\mathbf{r}}{|\mathbf{r}|^3}\,\mathrm{d}s\,\mathrm{d}s'. \tag{1}$$

Gauge invariance of H relies on the condition $B_n|_{\partial V} = 0$ often violated in solar physics. Berger and Field[1] showed how gauge invariance may be restored by measuring the helicity with respect to a reference magnetic field $\mathbf{B}'$ sharing the same distribution of B_n on ∂V. This relative helicity is then an ideal invariant under motions that vanish on ∂V. It has since been widely applied to the open magnetic fields arising in solar physics, but still requires the choice of an arbitrary reference field. Other authors have proposed alternative definitions of the magnetic helicity which certain desirable properties (see a discussion and references in [2]). None of the proposed open helicity definitions had previously been shown to be unambiguously preferable.

In [2] we showed each of these quantities (including the relative helicity) can be interpreted in terms of an open analogue of the linking number, the *net winding* $\mathcal{L}(\mathbf{x},\mathbf{y})$ of an angle Θ made by the two curves in a plane of fixed Cartesian z coordinate,

$$\mathcal{L}(\mathbf{x},\mathbf{y}) = \int_0^h \frac{\mathrm{d}\Theta}{\mathrm{d}z}\,\mathrm{d}z = \frac{1}{2\pi}\Big(\Theta(\mathbf{x}(h),\mathbf{y}(h)) - \Theta(\mathbf{x}(0),\mathbf{y}(0))\Big) + N, \tag{2}$$

with N a signed number of integer windings (see Figure 1(a)). This quantity is clearly invariant to all motions which vanish at the end planes. In general the curves will turn back on themselves and in order to maintain invariance $\mathcal{L}(\mathbf{x},\mathbf{y})$ the curve must be split into sections by its turning points and the winding of each individual section must be measured [4] (see Figure 1(b) and (c)).

Crucially it was demonstrated in [2] that all possible definitions of the magnetic helicity, save one particular choice of gauge, measure the rotation of Θ with respect to a moving basis, so that untwisted curves can be measured as entangled when they are not (see Figure 2). This additional rotation is defined entirely by a gauge transformation form the preferred gauge/reference field and hence has no physical interpretation.

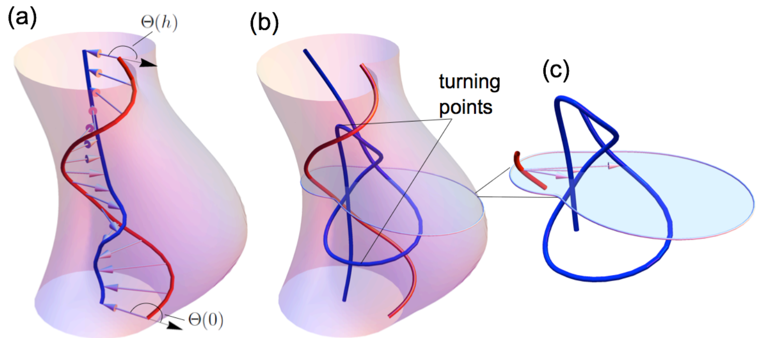

Fig. 1: Geometrical interpretation of the winding number.

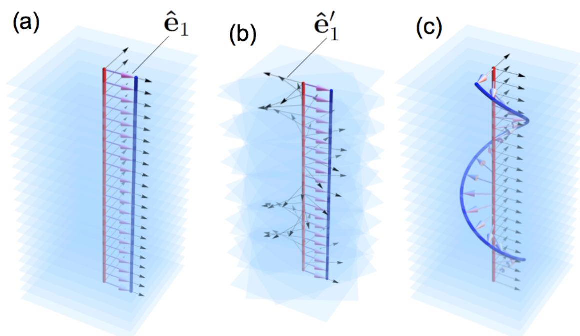

Fig. 2: How a straight field can appear twisted when measured with respect to a rotating frame. (c) the curves in (c) are wound equivalently to the straight curves of (a) as seen form a rotating basis (b)

We shall present these results in more detail and further show the quantity $\mathcal{L}(\mathbf{x}, \mathbf{y})$ can be used provide a complete topological invariant for braided magnetic fields. This allows us to accurately measure reconnection in braided fields In addition, if time permits we shall present some recent work applying this topological framework to experimental data of two reconnection magnetic flux ropes obtained form the UCLA basic plasma facility.

References

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