Submanifold helicities

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Summary

The helicity of a vector field is a measure of the average linking of pairs of integral curves of the field. The authors have previously translated this to the helicity of a differential form, dual to the vector field. This allows helicity to be viewed as a cohomology class on an appropriate manifold, the configuration space $C_2[\Omega]$; here Ω has the same dimension as the ambient space $\mathbb{R}^n$. We now consider the case where Ω has a lower dimension than the ambient space, and define *submanifold helicity* in this case. This submission represents an introduction to our work. We are able to show that the ambient space must have dimension $4i+3$, and that some submanifold helicities exhibit diffeomorphism invariance, while some do not.

1 Helicity via Cohomology Classes

The helicity of a magnetic field was first defined by Woltjer [5] in 1958 and named by Moffatt [3] in 1969. The helicity of a divergence-free vector field is invariant under volume-preserving diffeomorphisms which are homotopic to the identity. In our paper [1], we define helicity for closed differential forms and easily obtain this diffeomorphism invariance by viewing helicity as a cohomology class on the configuration space $C_2[\Omega]$. Here, $C_2(\Omega)$ is the configuration space of ordered pairs of points in the subdomain $\Omega^n \subset \mathbb{R}^n$, while $C_2[\Omega]$ is its Fulton-Macpherson compactification. See also [2, 4] for details on using configuration spaces to approach problems of knot and vector field invariance.

2 Submanifold Helicities

In our paper [1], we have only considered the case where Ω is a top-dimensional subdomain of $\mathbb{R}^{2k+1}$. We turn our attention now to the case where Ω has smaller dimension than the ambient space. We say α is a *Dirichlet (differential) form* on Ω if α vanishes on the boundary of Ω.

Definition 1. The *submanifold helicity* $\mathrm{H}_{n,m}$ is defined for closed Dirichlet $(k+1)$-forms on an n-dimensional submanifold Ω of $\mathbb{R}^m$ by

$$\mathrm{H}_{n,m}(\alpha) = \int_{C_2[\Omega]} \alpha_x \wedge \alpha_y \wedge g^* \,\mathrm{dvol}_{S^{m-1}} \,. \tag{1}$$

In order for the integrand $\Phi_m = \alpha_x \wedge \alpha_y \wedge g^* \,\mathrm{dvol}_{S^{m-1}}$ to be a $2n$-form, we require that $2k+2+m-1 = 2n$, i.e., that $k = n - (m+1)/2$. We will also refer to $\mathrm{H}_{n,m}$ as *(n,m)-helicity.*

Observe that the ambient space must have odd dimension m; we may consider submanifold dimensions $(m-1)/2 \leq n \leq m$. Below we show that all submanifold helicities are identically zero when $m = 4i+1$ and thus the only interesting results occur when m is of the form $4i+3$.

Example 2. Here are a few examples of submanifold helicities.

1. Submanifold helicities generalize the standard definition of helicity. Indeed, the $(3,3)$-helicity $\mathrm{H}_{3,3}$ represents standard helicity.
2. In [1], we extended the definition of helicity to $(k+1)$-forms defined on top-dimensional subdomains of $\mathbb{R}^{2k+1}$; this appears as $\mathrm{H}_{2k+1,2k+1}$.

3. The $(1,3)$-helicity $\mathrm{H}_{1,3}$ of 0-forms on a curve in $\mathbb{R}^3$ turns out to be precisely the writhing number of the curve, so we know that submanifold helicities are not always invariant under diffeomorphisms (since writhe is not).
4. The $(2,3)$-helicity $\mathrm{H}_{2,3}$ measures the linking of a 1-form on a surface in $\mathbb{R}^3$.
5. We do not know whether $\mathrm{H}_{5,7}$, which measures the linking of a 2-form on a 5-dimensional surface in $\mathbb{R}^7$, is an invariant.

Question. Two questions arise immediately. (1) Which submanifold helicities $\mathrm{H}_{n,m}$ are invariant under diffeomorphisms? (2) When is $\mathrm{H}_{n,m}$ nontrivial?

As for helicity, we know that submanifold helicity will be an invariant if the closed form $\alpha_x \wedge \alpha_y \wedge g^* \operatorname{dvol}_{S^{m-1}}$ is Dirichlet, i.e., if it vanishes on the boundary of $C_2[\Omega]$. Following arguments from [1], we only have to worry about the face (12) of this boundary, which is diffeomorphic to $\Omega \times S^{m-1}$. On this face, $\alpha_x \wedge \alpha_y$ pulls back to $\alpha \wedge \alpha$. Our previous argument depended on the observation that this was a $(2k+2)$-form $\alpha \wedge \alpha$ on the $n = (2k+1)$-manifold Ω. In general, $2k+2$ may not be greater than n, so we cannot depend on this argument. However, we note that if $k+1$ is odd, then $\alpha \wedge \alpha$ vanishes on Ω by antisymmetry, providing a partial answer to the first question above.

Proposition 3. *If $k+1 = n - (m-1)/2$ is odd, then $\mathrm{H}_{n,m}$ is invariant.*

What about the second question? For a contractible domain $\Omega = D^n$, the configuration space $C_2[\Omega]$ has the topology of $D^n \times D^n - \{\mathrm{pt}\} = D^{n+1} \times S^{n-1}$. In this case, only the cohomology groups $H^*(C_2[\Omega], \partial C_2[\Omega])$ where $* = 0, n+1, 2n$ are nontrivial. So for $H_{n,m}$ to be nontrivial in this case, we must have $2k+2 = n+1$, which only occurs for $\mathrm{H}_{2k+1,2k+1}$. But if Ω has nontrivial homology, then $C_2[\Omega]$ has more homology and (n,m)-helicity might be nontrivial. For example, we conjecture that if Ω has 1-dimensional homology, then the invariant $\mathrm{H}_{2,3}$ is nontrivial on Ω.

When $m = 4i+1$, the generalization of helicity must be identically zero, as we proved in [1]. The corresponding result holds for (n,m)-helicities.

Proposition 4. *If the ambient space has dimension $4i+1$, then $\mathrm{H}_{n,m}$ is identically zero for any $(k+1)$-form α.*

Proof. We consider the automorphism a of $C_2(\Omega)$ that interchanges x and y; it extends naturally to $C_2[\Omega]$. It changes the orientation of $C_2[\Omega]$ by a factor of $(-1)^{n^2} = (-1)^n$.

Next, we take the pullback $a^*\Phi_m = \alpha_y \wedge \alpha_x \wedge a^* g^* \operatorname{dvol}_{S^{m-1}}$. The map a induces an antipodal map on S^{m-1}; since m is odd, such a map has degree -1. Hence, $a^* g^* \operatorname{dvol}_{S^{m-1}} = -g^* \operatorname{dvol}_{S^{m-1}}$. Also, $\alpha_y \wedge \alpha_x = (-1)^{(k+1)^2} \alpha_x \wedge \alpha_y$. Combining these results, $a^*\Phi = (-1)^k \Phi$. We then compute

$$\int_{C_2[\Omega]} a^*\Phi = \int_{a(C_2[\Omega])} \Phi \qquad \Rightarrow \qquad (-1)^k \, \mathrm{H}_{n,m}(\alpha) = (-1)^n \, \mathrm{H}_{n,m}(\alpha) \tag{2}$$

For $m = 4i+1$, then k and n have the opposite parity, which implies that $\mathrm{H}_{n,m}(\alpha) = -\mathrm{H}_{n,m}(\alpha)$, i.e., that helicity is zero, and proves our proposition. For $m = 4i+3$, the conclusion is a tautology: $\mathrm{H}_{n,m}(\alpha) = \mathrm{H}_{n,m}(\alpha)$. □

References

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