Entanglement Transitions in Confined Fluid Flows

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Summary

Employing a course grained Periodic Boundary Condition (PBC) simulation, we consider the entanglement of fluid flow trajectories, vortex lines, confined to a tube with square cross section when subjected to flow alignment constraints. We study these as a function of the complexity of the system, the scale of the cross section, and the strength of the alignment constraint. To do so, we employ Panagioutous periodic linking number and periodic self-linking number, notions related to the helicity of the system, as well as the associated periodic linking matrix and the eigenvalues that are determined by this matrix. These provide physical measures of the structure and, thereby, insights into the resulting character of the entanglement in these models. As a consequence, we are able to determine the topological transitions associated to changes in the structural complexity of the system, the cross section scale, and the alignment constraint.

1 Introduction

In 1877, Gauss defined the linking number associated to two closed oriented smooth curves, $\mathbf{L_1}$ and $\mathbf{L_2}$, in space [3]:

$$lk(\mathbf{L_1}, \mathbf{L_2}) = \frac{1}{4\pi} \int_{\mathbf{L_1} \times \mathbf{L_2}} \frac{(\gamma_1(t) - \gamma_2(s)) \cdot dA}{\|\gamma_1(t) - \gamma_2(s)\|} \quad (1)$$

For a system of two line vortices , $\mathbf{L_1}$ and $\mathbf{L_2}$, of strengths $\kappa_\mathbf{1}$ and $\kappa_\mathbf{2}$, satisfying certain conditions, in 1969 Moffat defined the helicity in an appropriate spatial region, V, related to the linking number [1, 2]:

$$H = \int_V \vec{u} \cdot \vec{\omega}\ dV = 2\ lk(\mathbf{L_1}, \mathbf{L_2})\ \kappa_1\ \kappa_2 \quad (2)$$

establishing a relationship between the helicity and properties of a discrete course-grained simulation that can be employed in simulations such as the Periodic Boundary Condition (PBC) models we employ here, [4, 5].

Our PBC models are generated by a cell, with square cross-section, containing polygonal arcs with identical normal intersection on the two opposing faces or interior endpoints. This structure generates the infinite *periodic system* whose polygonal arcs are required, in the present case, to be finite. The *periodic linking number* between distinct arcs is definded to be the sum of the linking numbers of the first with all the translates (in the periodic system) of the second [5], Analogously, one may define the *periodic self-linking number* of an arc. These determine the *periodic linking matrix* consisting of entries determined by these periodic linking numbers and the resulting eigenvectors (since this is a real symmetric matrix) that expresses topological properties of the system. The constraints are realized by using unit length polygon edges, varying the edge length of the square cross-section to realize the confinement, and the degree of alignment of the directed edges with the positive axis of the system to control the character of the flow, evolving between *chaotic* and *strongly aligned*.

2 Structural Transitions

As the number and length of the vortex lines change, the degree of confinement measured by the dimension of the square cross-section, and the directional strength of the flow as measured by the strength of alignment to the axis, we report the evolution of the densitiy of the flow (measured by the squared radius of gyration of the vortex lines) and the evolution of the topoplogical entanglement (measured by the eigenvalues of the periodic linking matrix). This evolution reports the entanglement transitions in these course grained simulations of confined fluid flows.

References

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