Curvature instability in vortex rings and helical vortices

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Summary

We provide theoretical predictions for the growth rate of the curvature instability in a curved Batchelor vortex as a function of the axial velocity parameter, the local curvature and the Reynolds number. The results are validated by direct numerical results of the linearized equations. They are then applied to rings and helices and compared to recent experimental observations.

Most vortices observed in nature are curved. The local curvature of the vortex affects the displacement of the vortex structure but it also modifies its internal structure by generating dipolar corrections. Callegari & Ting [1] showed that these corrections can be calculated for any vortex model using asymptotic techniques when the vortex core size is small compared to the curvature radius. These corrections are responsible for the so-called curvature instability [2]. This short-wavelength instability develops in the core of curved vortices. It results from the resonant coupling of two Kelvin modes of the underlying straight vortex with the dipolar correction associated with curvature. This instability is similar to the elliptical instability for which the coupling occurs through quadripolar corrections (i.e. strain field) [3].

The curvature instability has been analyzed by Hattori & Fukumoto [2, 4] for a Rankine vortex (uniform vorticity in the vortex core). Such a vortex model is convenient as it allows a complete analytic treatment. However, it possesses properties which are not shared by smoother vortex profiles such as the Lamb-Oseen vortex (Gaussian vorticity). For the Rankine vortex, the Kelvin mode frequencies cover the entire range $((-2+m)\Omega_{\max}, (2+m)\Omega_{\max})$ for each azimuthal wavenumber m ($\Omega_{\max}$ being the maximum angular velocity of the Rankine vortex). This property is not satisfied by the Lamb-Oseen vortex which possesses a gap in its spectrum of Kelvin Modes. In particular, no regular Kelvin mode of azimuthal wavenumber $m > 0$ exists in the frequency range $(0, m\Omega_{\max})$: These modes would exhibit a singularity at the critical point r_c where the mode frequency ω satisfies $\omega = m\Omega(r_c)$. The main effect of this singularity is to damp the modes, with a damping rate which becomes important when the critical point gets close to the vortex core [5].

The curvature instability requires the existence of two Kelvin modes of azimuthal wavenumbers m and $m+1$ with the same frequency and axial wavenumber. For the Lamb-Oseen vortex, we show that this condition of resonance necessarily involves a singular Kelvin mode. The instability can therefore be present only if the growth associated with the resonant coupling can overcome the critical layer damping of the Kelvin mode. This is a strong constraint which selects very few possible resonant configurations. Only Kelvin modes of azimuthal wavenumbers $m = 0$ and $m = 1$ with a high radial complexity are found to be possible. However, the complexity of these modes makes them more sensible to viscous effects. We show that they are stable for the moderate Reynolds numbers considered in most vortex ring experiments, which could explain why they have never been observed.

When a Gaussian axial flow is present in the vortex core, the Lamb-Oseen vortex becomes the so-called Batchelor vortex. Such a model is usually used to describe tip vortices as generated by wings or rotor blades. In the present work, we have analyzed the effect of the amplitude of the axial flow (defined as $W_0 = 2\pi a^2 W_{\max}/\Gamma$ where $W_{\max}$ is the maximum axial velocity, Γ the vortex circulation, a the vortex core radius) on the curvature instability in the regime where the Batchelor vortex is not affected by the swirling jet instability [6], that is for $W_0 < 0.6$. We show that the curvature instability is possible in the presence of axial flow. We obtain that it still involves Kelvin modes of azimuthal wavenumbers $m = 0$ and $m = 1$ but their radial complexity becomes simpler as W_0 increases. When $W_0 = 0.5$, the most unstable mode resembles the elliptic instability mode developing in vortices without axial flow but its wavelength is almost twice larger. It could correspond to the short-wavelength instability mode recently observed in experiments by Leweke et al [7] in the helical vortex created by a single-blade rotor (see Fig. 1). The experimental visualization indeed shows a single strand structure with a wavelength $\lambda \approx 6a$ close to the theoretical prediction ($\lambda = 5.7a$).

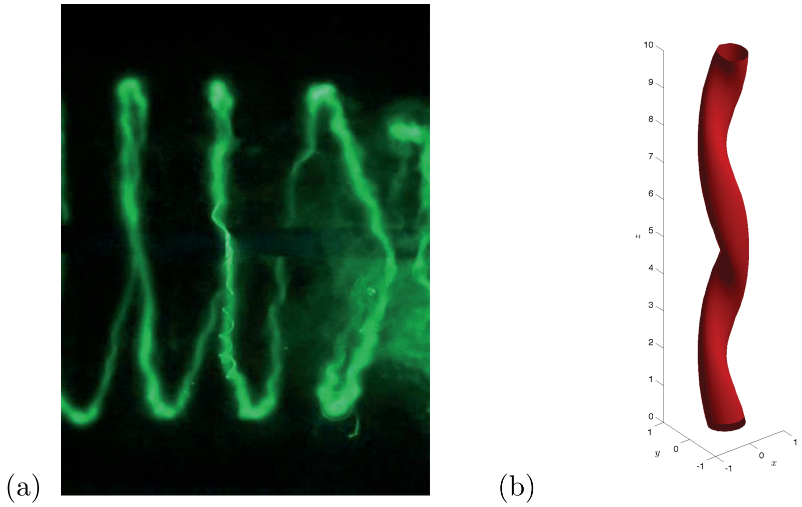

Fig. 1: (a) Visualisation of the short-wavelength instability developing in the core of a helical vortex (From T. Leweke, private communication). (b) Most unstable mode of the curvature instability for the experimental parameters of (a). Here is plotted an axial vorticity contour of the vortex deformed by the most unstable mode.

The theoretical results are also validated by performing direct numerical simulations of the equations linearized around the base flow formed by the Batchelor vortex plus its dipolar correction. Both the structure and the growth rate of the perturbations obtained for large time from white noise initial conditions are compared to the theory, and a good agreement is demonstrated.

Acknowledgements. Financial support from the french Agence Nationale de la Recherche (ANR Helix project ANR-12-BS09-0023-01; LABEX MEC Project ANR-11-LABX-0092) is kindly acknowledged.

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