Collision of point vortices

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Summary

The system of a finite number $n > 3$ of point vortices under suitable conditions can converged (collapse) to the point with finite time. It was described how to find the initial positions of vortices that collapsed. An explicit solution for collapsing trajectory was derived. The numerical evidence was presented that initial position of collapsing vortices organized their self in vortex sheets. Examples of the collapsing configurations with different number of vortices were presented.

1 INTRODUCTION

The collapse of the vortices belongs to one of the most interesting problems related to the dynamics of vortices. We will show that collapse vortices is possible for any number of the vortices, $n \geq 3$. The presence of the one or two strong vortices in the set of collapsing vortices lead to rising of the filaments created by point vortices. It can be interpret as vortex sheets [6].

2 EQUATIONS OF MOTION AND THEIR INVARIANTS

The equations of motion of the system of n-point vortices on the plane with distinct positions $\mathbf{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$, $z_k = x_k + iy_k$, and circulations $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$, each $\Gamma_j \in \mathbb{R} \setminus 0$ are

$$\frac{dz_k(t)}{dt} = \mathbf{v}_k(\mathbf{z}(t)) = \frac{i}{2\pi} \sum_{l=1, l\neq k}^{n} \Gamma_l \, \frac{z_k - z_l}{|z_k - z_l|^2} \tag{1}$$

It is well known that the systems (1) posses several invariants [1, 5, **?**]

$$\begin{gathered} M_x = \sum_k^n \Gamma_k x_k, \quad M_y = \sum_k^n \Gamma_k y_k, \quad S = \sum_k^n \Gamma_k (x_k^2 + y_k^2), \\ V = \frac{1}{2\pi} \sum_{k>j} \Gamma_k \Gamma_j, \quad H = -\frac{1}{4\pi} {\sum_{k,j}}' \Gamma_k \Gamma_j \ln r_{kj} \end{gathered} \tag{2}$$

3 SELF-SIMILAR MOTIONS AND COLLAPSE

Assuming the similarity motion of system of vortices $\frac{\mathrm{d}z_k}{\mathrm{d}t} = v_k = \lambda(t) z_k, \qquad k = 1, 2, \ldots, n$, one can check (see [5]) that $\mathbf{v}_1 z_k = \mathbf{v}_k z_1 \quad (k = 1, .., n-3)$. Collapsing trajectory are disrobed by the formula

$$z_k(t) = \sqrt{2\lambda_r(0)t + 1} e^{\left(i \frac{\lambda_i(0)}{2\lambda_r(0)} \ln(2\lambda_r(0)t+1)\right)} z_k(0) \tag{3}$$

The critical (collision) time $t \to T_c$ is $T_c = -\frac{1}{2\lambda_r(0)}$. The Hamiltonian H in(2) during the motion will be conserve, when the invariant V in (2) is equal zero, $V = 0$. Now one should find the $z_k(0)$. Without loosing the generality we assumed that $M_x = 0$ and $M_y = 0$.The collapsing positions of vortices can be determined by common zeros of the functions $V = 0$, $M_x = 0$, $M_y = 0$, $S = 0$ and $2(n-3)$, $f_j = \mathbf{v}_1 z_{j+2} - \mathbf{v}_{j+2} z_1, j = 2, \ldots, n-2$. To complete the systems to $2n$ equations, it was assumed that one of the vortex in the system e.g z_n has the fixed position, and it was included to the system of equation the identity $\sum_j \Gamma_j z_j^* \mathbf{v}_j = 0$ [3, 4]. The nonlinear algebraic system of equation was solved by the Newton method [3, 4].

4 NUMERICAL RESULTS

In Figures 1 A), B), C) were presented the cases with different number of stronger vortices in set of the collapsing 72 vortices. In Figure 1 A) intensities were $\Gamma_{1\ to\ 71} = 1$ and only one vortex had dominat value $\Gamma_{72} = -35$($T_c = 5.86232$, $H = 49.036$). In Figure 1 B) intensities were $\Gamma_{1\ to\ 70} = -1$ and two vortices had dominat value $\Gamma_{71,72} \approx 20.1502$,($T_c = 32.72233$, $H = 21.961502$) and in Figure 1 C) intensities were the same as in Figure 1 B) but now $H = 14.8442$ and collision time was very long $T_c \cong 63511.8$. Figure 1 B) there were two strong vortices $\Gamma_{71,72} \approx 20.1502$,and $\Gamma_{1\ to\ 70} = -1$. Collision time was $T_c = 32.72233$ and $H = 21.961502$. In Figure 1 C) there were also two strong vortices $\Gamma_{71,72} \approx 20.1502$, $\Gamma_{1\ to\ 70} = -1$, and $H = 14.8442$ but collision time was very long $T_c \cong 63511.8$. One can noticed (see Figure 1 C)) that vortices created the circle that took part in collapsing. Changing the Hamiltonian value in some intervals ($[H_{nim}, H_{max}]$) one can find the continuous collapsing curves where vortices lie. That curves originate (H_{nim}) and terminate H_{max} for points that are very closed to relative equilibrium ($T_c = \infty$).

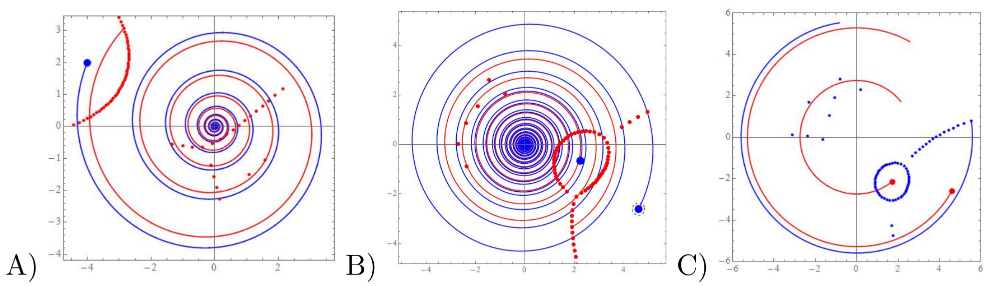

Fig. 1: Examples of collapsing systems of 72 vortices; In order to keep the readability of the graph it was shown only a few trajectories. Thick points mark the stronger vortices. The blue color related to negative intensity, the red color to the positive one.

5 CONCLUSIONS

From examples it is clear that the presence a few stronger vortices in the set of vortices for which we are looking the collapsing positions of vortices caused that the vortices accumulated in the pieces of curves that can be interpret as vortex sheets. In the atmospheric physics, it seems that collapsing vortices have relevance to the meteorological phenomenon of a polar cyclone [4]. Spirals and filaments of vorticity are characteristic feature of two-dimensional turbulent flows [2].

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