# What trefoil reconnection says about Navier-Stokes regularity

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**Summary**

Scheeler et al. [1] have recently demonstrated that one can experimentally investigate helicity by imprinting high Reynolds number helical vortex knots into a fluid by yanking 3D-printed aerofoils covered with hydrogen bubbles out of a water tank. Some surprising claims were made based upon the evolution of the resulting vortex filaments. This contribution will address those claims by simulating the evolution and self-reconnection of a similarly perturbed trefoil vortex knot over a range of Reynolds numbers and core diameters. The surprisingly long time it takes for reconnection to begin is confirmed. However, the simulations suggest that the significance of the trefoil it is not just in the initial preservation of helicity, but in how it is dissipated in a finite time once reconnection begins. This result implies that the trefoil's vortex dynamics is controlled primarily by the helicity, and not the energy, enstrophy or peak vorticity.

One of the fundamental, unanswered questions about turbulence is why every physical turbulent flow dissipates finite energy in a finite time. Which seems to be inconsistent with the best current mathematics, which indicates that no Navier-Stokes solution from any smooth initial condition can dissipate finite energy in a finite time as the viscosity goes to zero.

This presentation will use the unique properties of a trefoil vortex to address this paradox by taking a step back and asking what defines the evolution of the vortices in incompressible fluids. Is it the energy and enstrophy/dissipation? Or the topology and helicity of the configuration? And if it is the helicity, what is the limit as the diameter of the filament goes to zero?

The trefoil vortex knots that these simulations [2] and the earlier experiments have generated seem particularly well-suited for addressing these questions for the following reasons.

- First, the helicity of these simulations is roughly 1/4 the upper bound given by a single perfectly helical Fourier mode. The most helical I have seen.
- Second, to fully address the Navier-Stokes regularity problem one must have initial conditions whose major Sobolev and Lebesgue norms are finite in Whole (infinite) Space. Other initial conditions such as initially anti-parallel and orthogonal vortices do not have this property.
- Third, if one decreases the filament's diameter $a$ while holding its circulation $\Gamma$ fixed, its kinetic energy and enstrophy grow without bound, while the configuration's helicity $\mathcal{H} = \Gamma^2\mathcal{L}$ remains constant. $\Gamma$ is the circulation and the self-linking number is $\mathcal{L} = \mathcal{W} + \mathcal{T}$ (writhe+twist).

* These contrasting trends make it easy to see whether the timescales are governed by the circulation and helicity, or the energy, enstrophy and the maximum vorticity.

Two figures are given. Fig. 1 illustrates the contortions of the trefoil just as reconnection begins using a vorticity isosurface, three closed vortex loops and two particular points. One where reconnection is beginning and the other the maximum of vorticity. The closed trajectories allow one to make qualitative comparisons with the bubble trajectories in the experiments [1] and relate the topological changes to some recent theoretical work [3]. Which is that the topological changes up to this time are independent of the viscosity of the calculations and the thickness of the filament, there has been very little reconnection and the helicity has not changed. All consistent with the surprisingly long period of helicity preservation in the experiments.

Note that in a classical fluid, reconnection is not instantaneous, so one of the curves (green) follows its original trajectory, while the red and blue curves represent a portion that has reconnected. At this early time, following predictions [3], the red and blue together preserve the total linking number of the trefoil. By determining the writhe, twist and the self-linking directly, primarily using

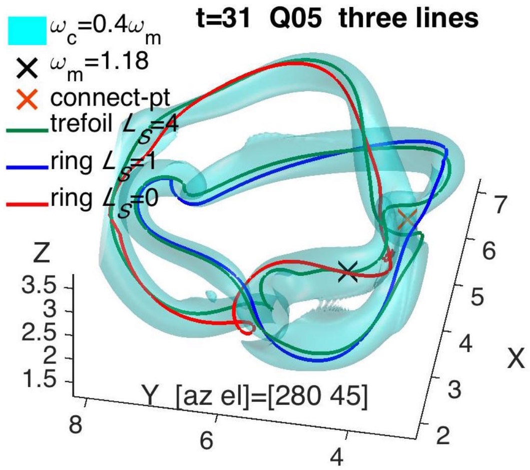


Fig. 1: A single vorticity isosurface plus three closed vortex lines at $t = 31$. The green line follows a remaining trefoil trajectories seeded near $\omega_m$, indicated by **X**. Its $\mathcal{L}_S = 4$, split into $\mathcal{W} + \mathcal{T} = 2.85 + 1.15 = 4$ The orange cross is the "reconnection point", the point between the closest approach of the trefoil's two loops and where, due to an extra twist, the loops are locally anti-parallel. The **Red** $\mathcal{L}_S = 0$ and **blue** $\mathcal{L}_S = 1$ lines originate on either side of the reconnection point and are linked, which gives a total linking of $\mathcal{L}_t = 2\mathcal{L}_{rb} + \mathcal{L}_{Sb} + \mathcal{L}_{Sr} = 2 + 1 + 0 = 3$, the linking of the original trefoil.

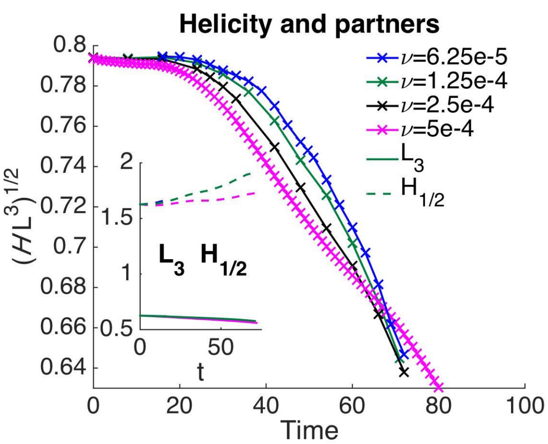


Fig. 2: Time evolution of the normalised helicity $(\mathcal{H}L)^{1/2}$ for 5 cases. Four with radius $a = 0.25$ for the viscosities listed, and one with radius $a = 0.175$ and $\nu = 0.0005$ that lies on the $\nu = 0.00025$ case shown. By $t \approx 72$ all cases have roughly the same decrease in helicity. The inset shows normalised $L_3$ and $H^{(1/2)} = \|u\|_{\dot{H}^{1/2}}$ for two of the calculations. $L_3$, $H^{(1/2)}$ and $\mathcal{H}$ are all normalised to have the units of circulation. $H^{(1/2)}$ must bound both $L_3$ and $|\mathcal{H}|$ from above and increases slowly, as required by its upper bound of $\sqrt{2EZ}$. None of which prevents the strong decrease in $\mathcal{H}$.

different forms of the Gauss-linking integral [2], it can be confirmed that the self-induced velocity along the filaments has three parts: The self-Biot-Savart term whose integral is the writhe, the traditional bi-normal self-induction, plus the torsion whose integral is the twist.

Fig. 2 illustrates how the helicity and related norms evolve in the period after reconnection begins for the five simulations [2] described in the caption. All dissipate approximately 1/3 of their helicity by $t = 65$. Over this period there is minimal energy loss, which decreases as the viscosity $\nu \to 0$. All three norms have been normalised to have the same units.

If $L_3 = \|u\|_{L^3} < \infty$ then Navier-Stokes is regular. Here $L_3$ is controlled as it decays just a bit more slowly than $L_2$ (2×energy). $|\mathcal{H}| \le H^{(1/2)}$ is an upper bound for helicity magnitude. Together these tell us why the energy spectrum [2] goes as $k^{-4}$. It is necessary for helicity dissipation. How $a = 0.175$, $\nu = 0.0005$ lays on the $a = 0.25$, $\nu = 0.00025$ case suggests that as $a \to 0$:

A. It is the helicity, not the energy, that controls the dynamics.

B. The time between when reconnection begins and ends goes to zero.

C. If seems possible that there can be finite helicity dissipation in a finite time.

* None of this violates any mathematics related to fixed smooth initial conditions.

What these calculations don't tell us is the conditions necessary for there to be finite energy dissipation in a finite time.

# References

[1] Scheeler, M. W., Kleckner, D., Proment, D., Kindlmann, G. L., & Irvine, W.T.M. (2014) *Helicity conservation by flow across scales in reconnecting vortex links and knots.* Proc. Nat. Acac. Sci. **111**, 1535015355.

[2] Kerr, R.M. (2015) *Simulated Navier-Stokes trefoil reconnection.* arXiv:1410.1259v1.

[3] Laing, C.E., Ricca, R.L. & Sumners, D.W.L (2015) *Conservation of writhe helicity under anti-parallel reconnection.* Sci. Rep. **5**, 9224.