The life of vortex knots and links and the conservation of helicity across scales

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Summary

Hydrodynamic helicity - a measure of the knottiness of vortex lines - is a conserved quantity in idealized fluids and thus offers the potential for fundamental insights into fluid flow. In real fluids, progress has been hindered by lack of accessible experimental and model systems. I will describe how to make vortex knots and links in water (in experiment), in the wave function of a superfluid (on a computer) and what happens thereafter. In particular, I will talk about how in helicity conservation, linking, coiling and twisting interplay across scales and types of flow. Finally, I will describe some progress towards exploring the effects of helicity on turbulent flows.

1 Hydrodynamic helicity

In addition to energy, momentum, and angular momentum, idealized (Euler) fluids have an additional conserved quantity, the hydrodynamic helicity $\mathcal{H}$:

$$\mathcal{H} = \int \mathbf{u} \cdot \omega \, \mathrm{d}V \tag{1}$$

where $\mathbf{u}$ and ω are respectively the velocity and vorticity of the flow. A natural interpretation of the hydrodynamic helicity is as a measure of the the linking and knotting of the vortex lines composing a flow. For an Euler fluid, the conservation of helicity follows from the Helmholtz laws of vortex motion, which forbid vortex lines from crossing and preserve the flux of vorticity, making it impossible for linked or knotted vortices to untie.

Conservation laws are of fundamental importance in understanding flows so the question of whether this topological conservation law extends to real, dissipative flows is of clear and considerable interest. Furthermore, since the dynamics of helicity are intrinsically geometric and topological in nature, lessons learned in fluids may well extend to the physics of tangling in other types of flow (e.g. plasmas) as well as other physical systems such as liquid crystals.

The robustness of helicity conservation in real fluids is unclear because dissipation allows the topology of field lines to change when nearby vortex tubes to "reconnect", apparently creating or destroying the topological linking of vortices. Analogous reconnection events have also been experimentally observed in superfluids and coronal loops of plasma on the surface of the sun and in general reconnection events exhibit divergent, nonlinear dynamics that makes it difficult to resolve helicity dynamics theoretically.

2 Generation of vortex knots

Thanks to a recent advance in experimental vortex production [1] it is now possible to generate vortices with controlled shape and topology, such as the one shown in Figure 1 (Left). The technique is based on the acceleration of 3D printed hydrofoils in an otherwise stationary fluid. Upon acceleration, a "starting vortex" whose shape traces the trailing edge of the hydrofoil is shed and subsequently evolves under its own influence. In our experiments, we use this technique to produce vortex knots and links in water having a total length of approximately $1m$, a typical width of 150 mm and circulation $\Gamma = 20,000 mm^2/s$; the Reynolds number is of order $Re \sim 2 \times 10^4$.

This enables us to *measure* the evolution of helicity in a viscous fluid and probe how well it is conserved as well as what mechanisms exist for its conservation, for example through reconnections.

3 Evolution of helicity

We find that in both cases, the vortices distort towards a series of localized reconnections that appear to change their coarse topology. To quantitatively measure the evolution of the helicity we consider the three

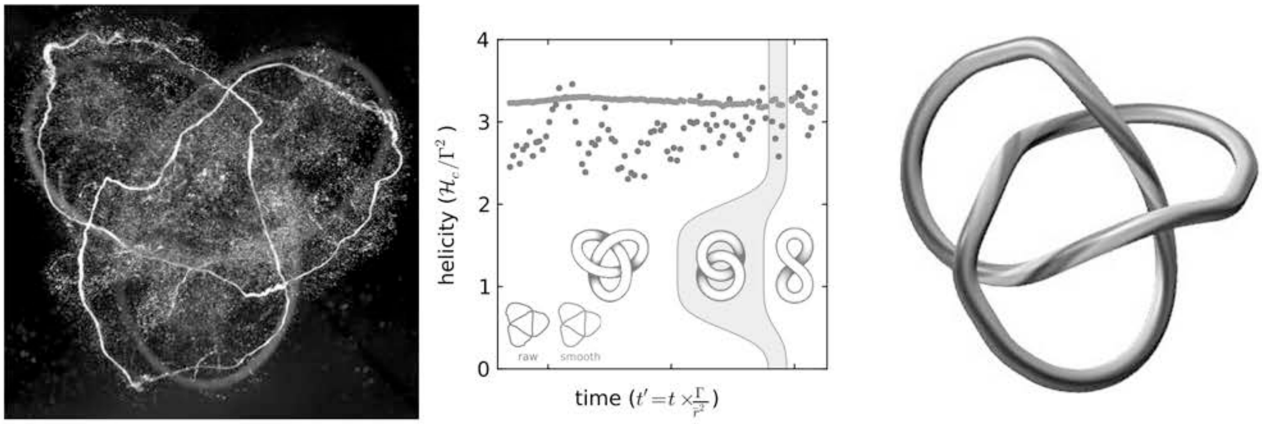

Fig. 1: (Left: A vortex knot in water. Middle: The evolution of link and writhe in a reconnecting vortex trefoil knot. Right: a vortex knot in the Gross Pitaevskii model of a superfluid.

geometrically distinct contributions to helicity for thin vortex tubes: Linking, coiling and twist. These three geometrically distinct but topologically equivalent contributions arise in the case of finite thickness vortex tubes, by subdividing the tube into N infinitesimal filaments each with strength $\Gamma = \Gamma_0/N$, and computing their linking in the limit $N \to \infty$:

$$\mathcal{H} = \sum_{i \neq j} \Gamma_i \Gamma_j \ \mathcal{L}_{ij} + \sum_i \Gamma_i^2 \ \left(Wr_i + Tw_i \right), \tag{2}$$

where L_{ij} is the linking number between tubes i and j, Wr_i is the writhe of the tube center-line, and Tw_i is the total twist of each tube.

Remarkably, we find that immediately following the reconnections, the loss of linking between vortex tubes is completely balanced by the presence of writhe in the reconnected tube centerlines (Figure 1, center). This occurs through a geometric mechanism that converts link to writhe. By further decomposing the writhe on different scales using a 'helistogram' we track the flow of helicity across scales [2].

The mechanism we find for helicity conservation through reconnections is entirely geometric, suggesting it may be present in other fluid-like systems as well. To test this possibility, we simulate the evolution of vortex knots in a superfluid with the Gross-Pitaevskii equation (GPE). Although superfluids are inviscid, they are not ideal Euler fluids: vortex reconnections are possible and vortex cores do not have unconstrained structure. By simulating vortices with hundreds of distinct topologies we find that both aspect of topological vortex evolution outlined so far are general: all vortex knots untie efficiently and as they reconnect conserve the helicity of their centerline [3].

Measuring the total helicity however, requires additional information about how the vortex lines are locally twisted inside the vortex core. To bridge this gap, we have developed a novel technique for experimentally measuring twist helicity. Using this method, we are able to measure the production and eventual decay of twist for a variety of vortex evolutions. Remarkably, we observe twist dynamics capable of conserving total helicity even in the presence of rapidly changing writhe. Injecting helicity in a turbulent flow (in experiment) enables us to explore the role helicity has to play in complex flows.

References

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