

# Variational formalism for linear growth rates of collisionless tearing modes

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## Summary

Growth rates of linear tearing instabilities in collisionless plasmas are investigated by formulating a variational principle in the form of Rayleigh quotient. In contrast to the conventional asymptotic matching, this method is basically applicable to static (no  $E \times B$  and diamagnetic drifts) and non-dissipative (purely collisionless) plasmas, but it can elucidate the general roles of the microscopic effects such as electron inertia and finite gyroradius, without specifying the equilibrium geometry or assuming the separations among the microscopic scales. The variational form is also useful from numerical point of view, in that the maximum growth rate can be calculated efficiently by using an iteration algorithm.

## 1 Outline

Magnetic reconnection in collisionless plasma is one of the phenomena that are being most actively researched in plasma physics. Since the magnetic reconnections in astrophysical and magnetically-confined plasmas are often too fast to be explained by the resistive diffusion process, other non-dissipative effects (such as electron inertia, Hall current, finite gyroradius and so on) should be taken into account for the purpose of extending the classical MHD framework. Although these effects are usually effective only in the neighborhood of the reconnection point, one has to choose a plausible model from a variety of two-fluid, gyrofluid, gyrokinetic equations, depending on how generally and how accurately one wants to incorporate the microscopic “nonideal MHD” effects. The fundamental problem is to understand the multiscale physics that causes fast magnetic reconnection, which remains challenging both theoretically and computationally.

The tearing instability is regarded as the trigger of magnetic reconnection. In both collisional and collisionless cases, the linear growth rate of the tearing mode is well investigated by using asymptotic matching expansion, in which the nonideal MHD effects are considered only inside the thin reconnection layer whereas the exterior region is solved by assuming the ideal MHD solution. This method is powerful, but it is generally difficult to see what is happening in the microscopic scales since the inner solution is solved in the Fourier space for a technical reason.

In this work, we attempt to formulate linear growth rates of tearing modes in collisionless plasmas as a variational problem (or a *Rayleigh quotient*). For ideal MHD instabilities, this variational formalism is widely known as the energy principle. In the same spirit, we show that the collisionless tearing instability can also admit the variational formalism by assuming the absence of *dissipation* (or collision) and *flow* including the  $E \times B$  and diamagnetic drifts. When this variational formulation is available, the eigenvalue problem is actually self-adjoint and the maximum growth rate can be evaluated efficiently by using an iteration algorithm for maximizing Rayleigh quotient. As is demonstrated below, the variational method is applicable not only to slab configuration (so-called Harris sheet) but also to any smooth equilibrium.

## 2 Variational formulation in a gyrofluid model

We assume the existence of a guide magnetic field  $B_0 = \text{const.}$  in the  $z$  direction and plasma is magnetized to it. When ion and electron temperatures are sufficiently low but the ion gyroradius  $\rho_i$  and the ion-sound gyroradius  $\rho_S$  are not negligible, the two-dimensional dynamics of plasma is

governed by a gyrofluid model [1, 2] for electrostatic potential  $\phi(x, y, t)$  and the  $z$  component of vector potential  $\psi(x, y, t)$  (see also [3, 4] for a detailed exposition);

$$\frac{\partial n_e}{\partial t} + [\phi, n_e] + [\nabla^2 \psi, \psi] = 0, \quad (1)$$

$$\frac{\partial \psi_e}{\partial t} + [\phi, \psi_e] - \rho_S^2 [n_e, \psi] = 0, \quad (2)$$

$$n_e - n_{i0} = \frac{\nabla^2}{1 - \rho_i^2 \nabla^2} \phi, \quad \psi_e = \psi - d_e^2 \nabla^2 \psi, \quad (3)$$

where a constant  $n_{i0}$  is the ion density, and  $d_e$  is the electron skin depth manifesting the effect of electron inertia. The neglect of all microscopic scales,  $\rho_i = \rho_S = d_e = 0$ , corresponds to the ideal MHD model.

Without the flow  $\phi \equiv 0$ , the only equilibrium condition is  $[\nabla^2 \psi, \psi] = 0$ , and it is solved by  $\nabla^2 \psi = f(\psi)$  with  $f : \mathbb{R} \mapsto \mathbb{R}$  being an arbitrary function. For smooth  $\psi$  such that  $|d_e^2 \nabla^2 \psi| \ll |\psi|$ , however, we may use an approximation  $\psi \simeq \psi_e$  to leading order. By denoting the linear perturbations by  $\tilde{\phi}, \tilde{\psi}, \tilde{\psi}_e$  and  $\tilde{n}_e$ , the linearized equations can be written in terms of a variable  $\lambda(x, y, t)$  that is defined by  $\tilde{\psi}_e = -[\lambda, \psi_e]$ . Then, the maximum growth rate  $\gamma$  is finally represented by a variational principle,

$$\gamma^2 = - \min_{\lambda} \frac{\delta W}{\delta I} := - \min_{\lambda} \frac{\int \left( [\lambda, \psi_e] \frac{-\nabla^2}{1 - d_e^2 \nabla^2} [\lambda, \psi_e] + [\lambda, \psi_e] [\lambda, \nabla^2 \psi] \right) dx dy}{\int \left( \lambda \frac{-\nabla^2}{1 - (\rho_i^2 + \rho_S^2) \nabla^2} \lambda \right) dx dy}. \quad (4)$$

Note that the denominator is positive definite ( $\delta I > 0$ ) and hence instability  $\gamma^2 > 0$  occurs if and only if there is a  $\lambda$  that makes  $\delta W$  negative. The first integrand in  $\delta W$  is also positive definite, which represents the stabilizing effect of the magnetic field-line's tension. It is clear that the role of the electron inertia is to weaken this stabilizing effect in a short wavelength scale  $\lesssim d_e$ . The collisionless tearing instability is understood as an instability  $\delta W < 0$  that is enabled by  $d_e \neq 0$  in spite of  $\delta W|_{d_e=0} > 0$  in the ideal MHD limit  $d_e = 0$ . For the case of  $\rho_S = \rho_i = 0$ , we have already proven  $\delta W < 0$  in a slab configuration by substituting a test function into  $\lambda$ , and estimated the growth rate which agrees with the asymptotic matching result [5]. Since  $\rho_i$  and  $\rho_S$  appear only in  $\delta I$  and an inequality  $\delta I|_{\rho_i=\rho_S=0} > \delta I$  holds, we can generally conclude that these gyroradii do not change the stability boundary but always enhance the growth rate.

More analytical and numerical results on the above and another two-fluid model will be given in the presentation.

**Acknowledgements.** This work was supported by JSPS KAKENHI Grant Number 25800308 and by JSPS Strategic Young Researcher Overseas Visits Program for Accelerating Brain Circulation # 55053270.

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