Time-dependent helical coordinates: derivation of the fundamental system and new conservation laws IUTAM Symposium, Venice 2016

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Summary

The current work is devided into two parts. In the first part we derived a reduced system of time-dependent helical invariant Euler and Navier-Stokes equations. By employing Lie Symmetry analysis, the spatial dependence of all independent variables is reduced by one and the remaining variables are: the cylindrical radius r and the helical variable $\xi = \frac{z}{\alpha(t)} + b\varphi$, $b = const.$ and time t. Assuming $\alpha = const.$, we retain the classical helically symmetric case discussed in [1]. Further, setting the parameters $b = 0, \alpha = 1$, we obtain the well known axisymmetric case, while $b = 1$, $1/\alpha = 0$ corresponds to a plane flow.

The reduction was done in three steps: In a first step we developed a new *helical, time-dependent* coordinate system. In the second step we expressed the Euler and Navier-Stokes equations in the new coordinate system. Therefor the dependent variables (the velocities and the pressure) and the derivatives were expressed through helical coordinates. Finally, helical invariance, i.e. $\frac{\partial}{\partial \eta} \equiv 0$ was imposed, leading to a helical invariant system of equations. This has been done both for primitive variables as well as for the vorticity formulation.

In the second part we sought new conservation laws which can be found from the helical invariant Euler and Navier-Stokes equations derived in the first part. The conservation laws were generated with the help of the GEM-tool [2] that is included into MAPLE. From this, we derived a variety of new conservation laws, mostly extensions of existing conservation laws, while certain classical conservation laws do not admit extensions in the time dependent frame.

1 Fundamental system and governing equations

To develop the new helical coordinate system, we considered the helical symmetry of the Navier-Stokes equations, which relies on the combination of two common Lie Symmetry groups: (i) generalized Galilean invariance with comprises classical Galilean group and axial translation and (ii) the rotation about the same axis. In Cylindrical coordinates the infinitesimal generators are given by

$$X_R = \frac{\partial}{\partial \varphi}, \quad X_T = \alpha(t)\frac{\partial}{\partial z} + \dot{\alpha}(t)\frac{\partial}{\partial u^z} - z\ddot{\alpha}(t)\frac{\partial}{\partial p}, \tag{1}$$

where $\alpha(t)$ is an arbitrary time-dependent function.

Considering the linear combination

$$X = \frac{1}{b}X_R - X_T \tag{2}$$

and applying the method of similarity variables, we obtain the new helical coordinates

$$\eta = b\varphi, \quad \xi = b\varphi + \frac{z}{\alpha(t)}, \quad \tilde{r} = r \tag{3}$$

in terms of Cylindrical coordinates r, φ, z.

Further, the helical velocity components and the modified pressure are given by

$$u^\xi = \left(\frac{b}{r}u^\varphi + \frac{1}{\alpha}(u^z + \dot{\alpha}b\varphi)\right) \cdot B(r,t) \tag{4}$$

$$u^\eta = \left(\frac{1}{\alpha}u^\varphi - \frac{b}{r}(u^z + \dot{\alpha}b\varphi)\right) \cdot B(r,t), \tag{5}$$

$$\tilde{u}^r = u^r, \qquad \tilde{p} = p + \frac{1}{2}\frac{\ddot{\alpha}}{\alpha}z^2, \tag{6}$$

where $B(r,t)$ is a form function and given by $B(r,t) = \frac{r\alpha(t)}{\sqrt{r^2+b^2\alpha(t)^2}}$.

A transformation of the Euler- and Navier-Stokes equations into the helical coordinate system and imposing helical invariance, i.e. $\frac{\partial}{\partial\eta} \equiv 0$, leads to a system of helically invariant equations that describe helical flows involving an arbitrary time-dependent parameter function $\alpha(t)$. Hence, we may compute helically invariant flows with a time dependent pitch. This is illustrated in figure 1.

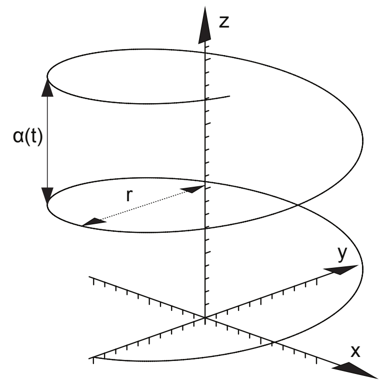

Fig. 1: An illustration of the helix ξ=const. with parameter function $\alpha(t)$ (cf. [1]).

2 New conservation laws for helical flows

In this second part we sought new local conservation laws of the form

$$\partial_t\Theta + \nabla \cdot \mathbf{\Phi} = 0, \tag{7}$$

where $\nabla \cdot \mathbf{\Phi} = \partial_i\Phi^i = \partial_1\Phi^1 + \partial_2\Phi^2 + \ldots + \partial_{n-1}\Phi^{n-1}$ is the spatial divergence. The quantity Θ is called density, whereas Φ^i are the spatial fluxes of the conservation law. To derive new sets of conservation laws we used the direct construction method [3]. We finally obtained multiple new local conservation laws, involving families of conservation laws, which are extensions of the classical case [1] wherein the parameter function α was assumed to be constant.

In primitive variables, one of the derived conservation laws is, e.g., an extension of the conservation of the z-projection of angular momentum:

$$\Theta = \alpha r B\left(\frac{1}{\alpha}u^\eta + \frac{b}{r}u^\xi\right) = \alpha r u^\varphi, \quad \Phi^r = \alpha r u^r u^\varphi, \quad \Phi^\xi = \alpha\left(r u^\xi u^\varphi + bBp\right) - \dot{\alpha} r \xi B u^\varphi. \tag{8}$$

We further obtained extensions of the conservation of generalized momenta and angular momenta and vorticity based conservation laws.

References

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