Atypical late-time singular regimes accurately diagnosed in stagnation-point-type helical solutions of 3D Euler flows

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Summary

We revisit numerically and analytically the finite-time blowup of an infinite-energy helical solution of 3D Euler equations [Gibbon et al. (1999)]. By employing the method of mapping to regular systems [Bustamante (2011), Mulungye et al. (2015)], we establish a curious property of this solution that was not observed previously: near singularity time T^*, a fast transient is followed by a slower late-time blowup regime that is well resolved spectrally at mid-resolutions (512^2), with a Gaussian wavenumber spectrum. The analyticity-strip width decays 'slowly' to zero at $t = T^*$, remaining above the collocation-point scale for all simulation times $t < T^* - 10^{-9000}$. Reaching such a proximity to singularity time is not possible in the original temporal variable, because of the floating-point double-precision barrier ($\approx 10^{-16}$). Due to this limitation on the *original* variables, the mapped variables now provide an improved assessment of the relevant blowup quantities, crucially with acceptable accuracy at an unprecedented closeness to singularity time: $T^* - t \approx 10^{-140}$.

1 Symmetry plane model and mapping to regular systems

We consider a class of exact solutions of the 3D Euler equations, possessing non-trivial helicity, presented by Gibbon *et al.* [1]. Writing $\mathbf{u}(x,y,z,t) = (u_x(x,y,t), u_y(x,y,t), z\,\gamma(x,y,t))$ we obtain

$$\frac{\partial \gamma}{\partial t} + \mathbf{u}_{\rm h}\cdot\nabla_{\rm h}\gamma = 2\langle\gamma^2\rangle - \gamma^2\,, \qquad \frac{\partial \omega}{\partial t} + \mathbf{u}_{\rm h}\cdot\nabla_{\rm h}\omega = \gamma\,\omega\,, \tag{1}$$

where $\mathbf{u}_{\rm h}(x,y,t) \equiv (u_x(x,y,t), u_y(x,y,t))$ denotes the "horizontal" component of the velocity field at the symmetry plane ($z = 0$), $\nabla_{\rm h} = (\partial_x, \partial_y)$ denotes the horizontal gradient operator, $\omega(x,y,t) = \partial_x u_y - \partial_y u_x$, is the vorticity, γ is the stretching-rate of vorticity, which using the incompressibility condition can be defined as $\gamma(x,y,t) = -\nabla_{\rm h}\cdot\mathbf{u}_{\rm h}(x,y,t)\,$, and $\langle f(\cdot,t)\rangle \equiv \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(x,y,t){\rm d}x{\rm d}y$ denotes the spatial average over the periodic 2D domain.

Analytic solutions are found for supremum norms by solving along characteristics [2, 3], leading to singular behaviour controlled by a classical Beale-Kato-Majda (BKM) type of theorem: $\int_0^T \|\gamma(\cdot,t)\|_\infty {\rm d}t < \infty$ for bounded solutions [4]. With the initial condition $\gamma_0(x,y) = \omega_0(x,y) = \sin(x)\sin(y)$, the singularity time is $T^* = \frac{4}{\pi^2}\int_0^1 \left[K(S^2)\right]^2\, dS \approx 1.4180027349238588750622 34$, where K is the complete elliptic function of the first kind. We can identify $\|\gamma(\cdot,t)\|_\infty = -\inf\gamma(\cdot,t)$.

Bustamante [5] and later Mulungye et al. [6] presented a method to bijectively map system (1) to one that is **globally regular** in time. Such a mapping removes some of the ambiguity of numerical assessment of singular behaviour and can increase the accuracy of important quantities. The BKM theorem provides a new 'mapped' time and a rescaling to mapped variables

$$\tau(t) = \int_0^t \|\gamma(\cdot,t')\|_\infty {\rm d}t', \qquad \gamma_{\rm map}(x,y,\tau) = \frac{\gamma(x,y,t)}{\|\gamma(\cdot,t)\|_\infty}\,, \qquad \omega_{\rm map}(x,y,\tau) = \frac{\omega(x,y,t)}{\|\gamma(\cdot,t)\|_\infty}\,, \tag{2}$$

resulting in the mapped version of system (1), with globally regular solution in mapped time τ:

$$\frac{\partial \gamma_{\rm map}}{\partial \tau} + \mathbf{u}_{\rm map}\cdot\nabla\gamma_{\rm map} = 2\langle\gamma_{\rm map}^2\rangle - \gamma_{\rm map}^2 - \gamma_{\rm map}\left\{1 - 2\langle\gamma_{\rm map}^2\rangle\right\}\,, \tag{3}$$

$$\frac{\partial \omega_{\rm map}}{\partial \tau} + \mathbf{u}_{\rm map}\cdot\nabla\omega_{\rm map} = \gamma_{\rm map}\,\omega_{\rm map} - \omega_{\rm map}\left\{1 - 2\langle\gamma_{\rm map}^2\rangle\right\}\,. \tag{4}$$

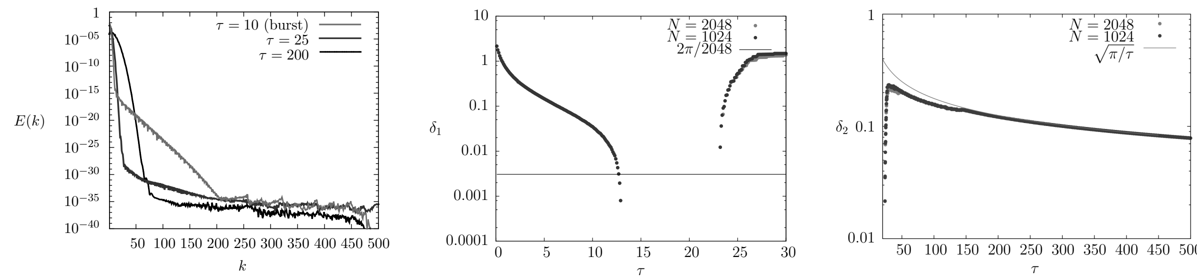

Fig. 1: γ^2-Fourier spectra snapshots (left) and time evolution of the analyticity-strip width $\delta(t(\tau))$. Centre: early-time profile from $E(k,t) \lessapprox C(t)k^{-n(t)}\mathrm{e}^{-2\delta_1(t)k}$, showing the initial burst. Right: late-time Gaussian profile from $E(k,t) \lessapprox C(t)k^{-n(t)}\mathrm{e}^{-(\delta_2(t)k)^2}$, showing the slow cascade and the rigorous estimate $\delta \approx \sqrt{\pi/\tau}$.

2 Diagnosis of singularity

We solve the evolution equations for both systems numerically using a Hou-dealiased pseudospectral method implemented on GPUs using CUDA [6], with a 4th-order Runge-Kutta time solver. Adaptive time stepping ($dt = d\tau/\|\gamma(\cdot,t)\|_\infty$) is used for the original equations and uniform steps of $d\tau$ are used in the mapped system with the resulting distribution of temporal data roughly equivalent.

The evolution of the circular-shell spectrum of stretching rate, $E(k,t) = \sum_{k-\frac{1}{2}<|\mathbf{k}|<k+\frac{1}{2}} |\hat{\gamma}(\mathbf{k},t)|^2$, shows two timescales in evidence. An initial burst can be observed, with a flux towards intermediate k which is redistributed across the modes and has the 'typical' spectrum $E(k,t) \lessapprox C(t)k^{-n(t)}\mathrm{e}^{-2\delta_1(t)k}$ [7, 3]. Provided $N > 256$ (spatial resolution) this initial phase remains well resolved and lasts only until $\tau \approx 25$ (in original variables this is $T^* - t \approx 10^{-10}$). Fig. 1 (centre) shows two resolutions ($N = 1024$ and 2048) which are essentially indistinguishable; negative values of δ_1 indicate that this ansatz ceases to be a valid analyticity measure due to the presence of a large scale Gaussian spectrum. The true late-time behaviour ($\tau > 25$) consists of a slow cascade that builds up from small k, with an 'atypical' spectrum $E(k,t) \lessapprox C(t)k^{-n(t)}\mathrm{e}^{-(\delta_2(t)k)^2}$ [3]. Fig. 1 (right) shows the slow decay of δ_2. This has been missed in previous work [7] as it only arises after the initial burst, and persists to sufficiently close to T^* to render it next to inaccessible without the mapped variables.

Errors in quantities such as stretching-rate norm $\|\gamma(\cdot,t)\|_\infty$ and singularity-time proximity $T^* - t$ (figures not shown), show that the original system's numerical solution loses accuracy beyond $\tau = 34$, corresponding to $T^* - t \approx 10^{-14}$ ('double-precision barrier'), while the mapped system's solution keeps a relative error of 10^{-7} beyond $\tau = 330$, corresponding to $T^* - t \approx 10^{-140}$. See details in [3].

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