Helicity of a small patch of quantum vorticity in superfluid helium

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Summary

We numerically study the evolution of a small patch of quantised vorticity in superfluid helium, and track the helicity of the vortex configuration as a function of time.

1 Introduction

Tangled filamentary structures occur in many physical systems, from ropes to DNA. In this work we are concerned with more 'mathematical' filamentary structures consisting of field lines in fluids and plasmas, for example vortex lines and magnetic field lines. Such lines undergo reconnection events, which tend to be associated with energy losses. In the limit of no dissipation, the governing equations of motion (the Euler equation, and the magnetic induction equation in the frozen field approximation) preserve the topology of the lines. Under this approximation, helicity and magnetic helicity are conserved quantities. Recent work suggests that in the case of small dissipation helicity is partially preserved [1]. The aim of this work is to explore this partial preservation of topology in a context where vortex lines are not mathematical abstractions but have a real physical meaning: superfluids such as liquid helium (^{4}He and ^{3}He) and atomic Bose-Einstein condensates.

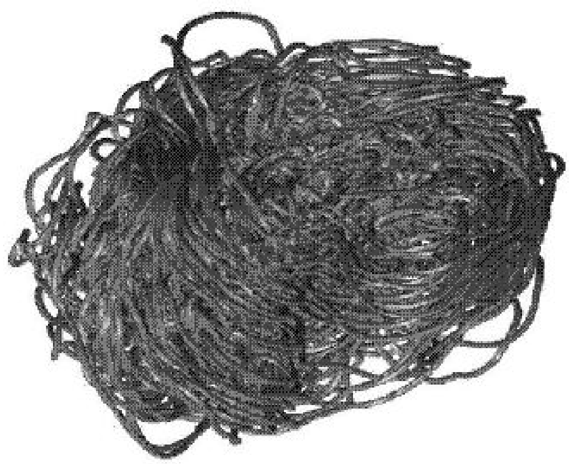

Fig. 1: Turbulent tangle of superfluid vortex lines.

2 Method

Superfluids are quantum fluids with zero viscosity and are governed by a macroscopic wavefunction $\Psi(\mathbf{x}, t) = \sqrt{n(\mathbf{x}, t)} e^{i\phi(\mathbf{x},t)}$ where $\mathbf{x}$ is the position, t is time, $\phi(\mathbf{x}, t)$ is the phase and $n(\mathbf{x}, t)$ the number density of atoms. Quantum mechanical prescriptions require that the velocity field is

$$\mathbf{v}(\mathbf{x}, t) = \frac{\hbar}{m} \nabla \phi(\mathbf{x}, t), \tag{1}$$

where $\hbar = h/(2\pi)$, h is Planck's constant and m the mass of one atom. Vorticity is thus either zero or takes the form of thin filaments around which the circulation is fixed:

$$\oint_C \mathbf{v} \cdot \mathbf{dr} = \kappa, \tag{2}$$

where C is an integration path around the axis of the vortex and $\kappa = h/m$ is the quantum of circulation. The vortex core is a small tubular region of radius $a_0 \approx 10^{-10}$ m (in ^{4}He), where the density goes to zero; in other words, although the velocity field around the vortex line, from Eq. (2), is $v = \kappa/(2\pi r)$ where r is the distance from the vortex axis, there are no atoms which move with infinite speed. Since in typical experiments a_0 is orders of magnitude less than the typical distance $\ell \approx 10^{-4}$ to 10^{-2} m between vortices, we model the vortex lines as space curves $\mathbf{s}(\xi, t)$ (where ξ is arc length) of infinitesimal thickness which move according to [2]:

$$\frac{d\mathbf{s}}{dt} = \mathbf{v}_{self} + \mathbf{w}, \tag{3}$$

$$\mathbf{v}_{self} = \frac{\kappa}{4\pi} \oint_{\mathcal{L}} \frac{(\mathbf{s}_1 - \mathbf{s}) \times \mathbf{s}_1}{|\mathbf{s}_1 - \mathbf{s}|^3}, \tag{4}$$

$$\mathbf{w} = \alpha \mathbf{s}' \times \mathbf{v}_{ns} - \alpha' \mathbf{s}' \times [\mathbf{s}' \times \mathbf{v}_{ns}], \qquad \mathbf{v}_{ns} = \mathbf{v}_n - \mathbf{v}_{self}, \tag{5}$$

Here $\mathbf{v}_n$ is the normal fluid and α and α' are small temperature-dependent friction coefficients arising from the interaction of vortex lines with the thermal excitations which make up the normal fluid. When numerically solving the above equations, we desingularise the Biot-Savart integral (4) using the vortex core radius a_0, and, following evidence from a more microscopic model [3], we algorithmically reconnect two vortex lines which are sufficiently close to each others. Note that in the zero temperature limit the normal fluid is negligible, α and α' vanish, and the vortices move simply as $d\mathbf{s}/dt = \mathbf{v}_{self}$.

3 Results

We start from a small number of randomly oriented vortex loop which serve as arbitrary initial condition. A localised region of normal fluid turbulence $\mathbf{v}_n$ (represented by a sum of random waves), models a local disturbance which can be be easily created in the experiments, and feeds energy into the vortex loops. The vortex loops become distorted, reconnect with each others and grow in length, until a statistical steady state is obtained - see Fig. (1) - in which the vortex length fluctuates around an average value.

During the time evolution, by numerically determining appropriate crossing numbers over given projections, we determine the linking number L_{ij} between any two loops i and j, and the (solid-angle averaged) writhing number W_i of each loop. We report the time behaviour of the helicity, defined as

$$H = \sum_{i \neq j} L_{ij} + \sum_i W_i, \tag{6}$$

and compare H to energy and length of the vortex configuration.

References

[1] Kleckner, D., Kauffman, L. H., & Irvine, T. M. How superfluid knots untie. arXiv:1507.07579 (2015).

[2] Schwarz, K. W., Three-dimensional vortex dynamics in superfluid ^{4}He: homogeneous superfluid turbulence. Phys. Rev. B **38**, 2398 (1988).

[3] Koplik, J. and Levine, H., Vortex reconnection in superfluid helium, Phys. Rev. Letters **71**, 1375 (1993).